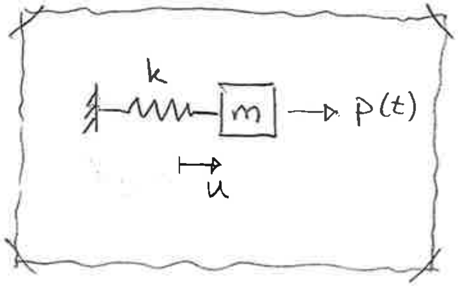
STRUCTURAL DYNAMIC COMPUTING

SDOF systems no damping:

- SDOF systems - equation of motion
- Free vibrations
- Forced harmonic vibration
- Dynamic response factor

THE UNDAMPED SDOF MASS-SPRING SYSTEM



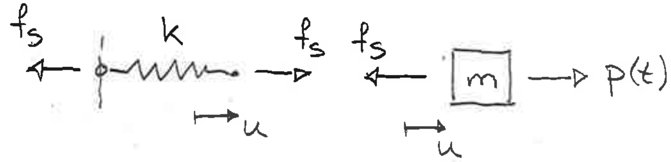
m : mass

k : spring konst. (N/m)

u : mass displacement

$p(t)$: external force

Free body diagram and Newton's second law:



$$\begin{cases} \text{spring force}: & f_s = ku & \ldots (1) \\ \text{Newton II (mass)}: & p(t) - f_s = m\ddot{u} & \ldots (2) \end{cases}$$

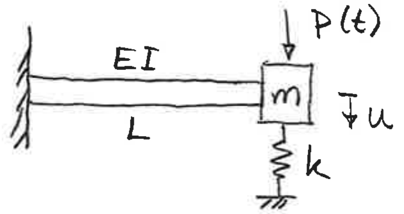
(1) into (2), (eliminate f_s) $\Rightarrow$

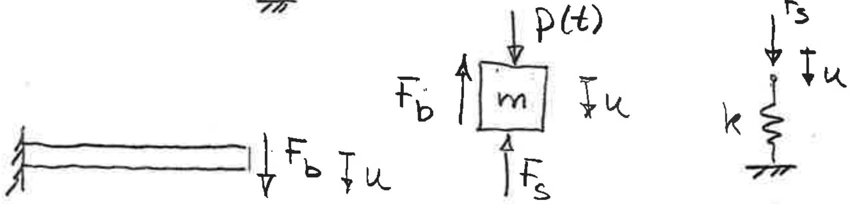
$$\boxed{m\ddot{u} + ku = p(t)} \quad \ldots (3)$$

The equation of motion (undamped)

Ex. SDOF mass and spring system

Beam with mass and spring:





Beam: $u = \dfrac{F_b L^3}{3EI}$; $F_b = \dfrac{3EI}{L^3} u$ ... (1)

Spring: $F_s = ku$ ... (2)

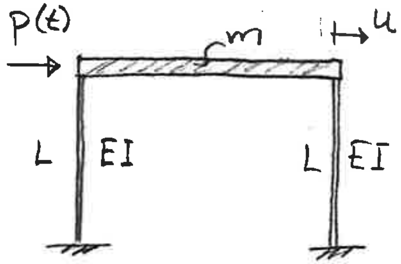
Mass: $p(t) - F_b - F_s = m\ddot{u}$ ... (3)

(1) and (2) into (3) ⇒

$$m\ddot{u} + \left(k + \frac{3EI}{L^3}\right) u = p(t)$$

The equation of motion #

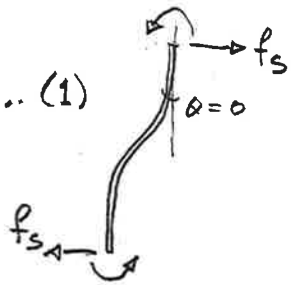
Ex. Shearbuilding – one floor



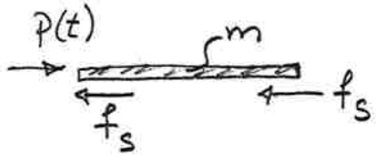
Rigid floor and light columns

Columns: $f_s = \dfrac{12EI}{L^3}$ ... (1)

(f_s from beam table elementary cases)



Mass:



NII (→) $p - 2f_s = m\ddot{u}$ ... (2)

Insert f_s from (1) into (2) ⇒

$$m\ddot{u} + \underbrace{\frac{24EI}{L^3}}_{k} u = p(t)$$

k "spring" stiffness in this case

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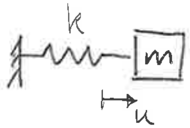
UNDAMPED FREE VIBRATIONS

SDOF system without external load:



The mass is given a velocity initally

The equation of motion:

$$m\ddot{u} + ku = 0 \qquad \dots (1)$$

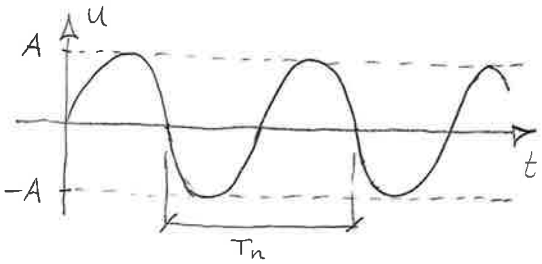
Trial solution: $\begin{cases} u = A \sin \omega t \\ \ddot{u} = -A\omega^2 \sin \omega t \end{cases}$ in (1) $\Rightarrow$

$$(-\omega^2 m + k) A \sin \omega t = 0 \quad \Rightarrow \quad \omega^2 = \frac{k}{m}$$

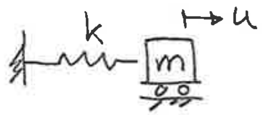
The natural circular frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$

with $\omega_n = 2\pi f_n$ and $f_n = \frac{1}{T_n}$



GENERAL SOLUTION – INITIAL VALUES – SDOF UNDAMPED SYSTEM – FREE VIB.



Initial values: $\begin{cases} u(0) = u_0 \\ \dot{u}(0) = v_0 \end{cases}$

Both $\sin \omega_n t$ and $\cos \omega_n t$ satisfy the equation of motion:

$$m\ddot{u} + ku = 0$$

The general solution is

$$u = A \cos \omega_n t + B \sin \omega_n t$$

A and B can be determined from the initial values

$u(0) = u_0 \Rightarrow A = u_0$

$\dot{u}(t) = -A\omega_n \sin \omega_n t + B\omega_n \cos \omega_n t$

$\dot{u}(0) = v_0 \Rightarrow B\omega_n = v_0 \; ; \quad B = \frac{v_0}{\omega_n}$

Thus

$$u(t) = u_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

is the general solution to the inital value problem

Ex. Mass and spring system at rest hit by a hammer.

Hammer:
Mass M and initial velocity V_0

Assume for simplicity that the hammer velocity after impact is zero.

Conservation of momentum:

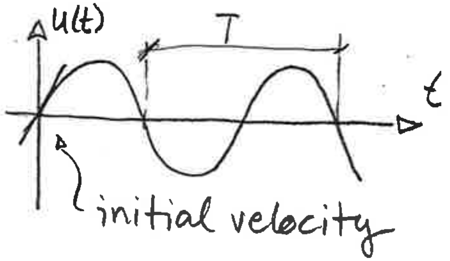
$(\leftarrow) \quad M V_0 = m \dot{u}(0) \; ; \quad \dot{u}(0) = \frac{M}{m} V_0$

General solution with $u(0)=0$

$$\begin{cases} u(0) = 0 \\ \dot{u}(0) = \frac{M}{m} V_0 \end{cases} \Rightarrow u(t) = \frac{M}{m}\frac{V_0}{\omega_n} \sin \omega_n t$$

$\omega_n = \sqrt{\frac{k}{m}}$

$\omega_n = \frac{2\pi}{T}$



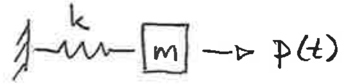
Rem. Coefficient of restitution

$$e = \frac{\dot{u}(0)}{V_0} = \frac{M}{m} \text{ required to}$$

stop the hammer $\Rightarrow M \leq m$

FORCED UNDAMPED HARMONIC VIBRATIONS

SDOF system



Eq.: $\quad m\ddot{u} + ku = p(t)$

Look at harmonic load: $p(t) = p_0 \sin \omega t$

Trial solution: $\quad u = u_0 \sin \omega t \quad \Rightarrow$

$(-\omega^2 m + k)\, u_0 \sin \omega t = p_0 \sin \omega t \;;$

$(k - \omega^2 m)\, u_0 = p_0 \;; \quad \boxed{u_0 = \frac{p_0}{k} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}}$

with $\omega_n = \sqrt{\frac{k}{m}}$

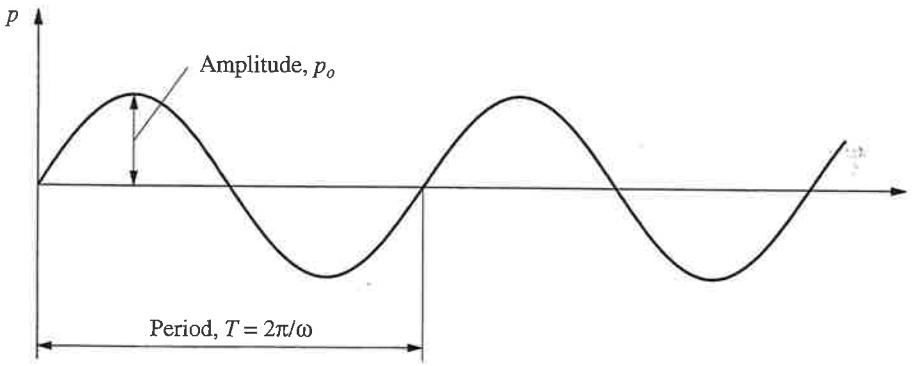
$\frac{p_0}{k}$ is the static solution with force p_0

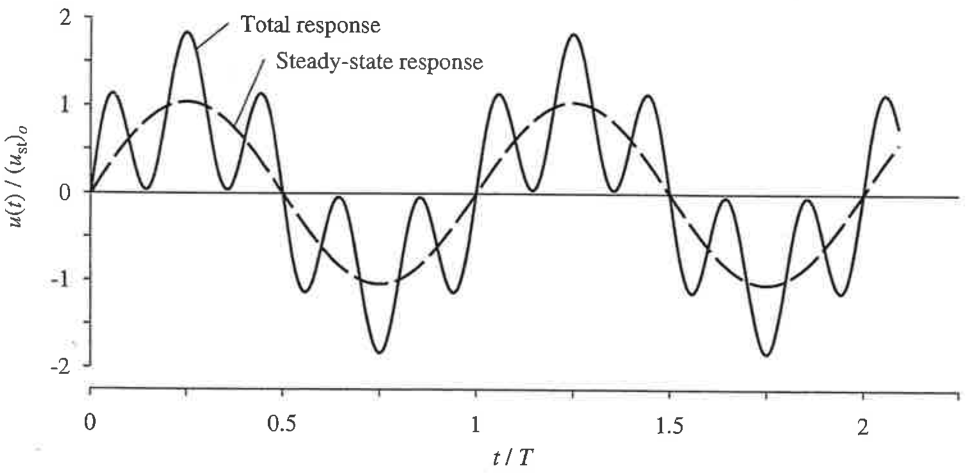
$\frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|} = R_d$ displacement response factor: $\frac{u_0}{u_{0\,stat}}$

Note! $\quad \omega \to \omega_n \Rightarrow u_0 \to \infty$ Resonance!

$\boxed{\text{$\omega = 2\pi f$ the steady state response frequency is equal to the loading frequency!}}$

The natural frequency is also triggered:

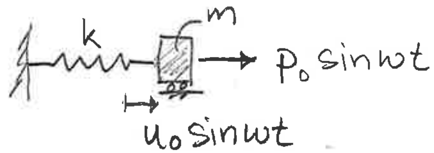




However, in real systems some damping is always present and only the steady state solution is left.

FORCED HARMONIC VIBRATIONS - DEFORMATION RESPONSE FACTOR - UNDAMPED

Steady state solution for the vibration amplitude:



$$\boxed{u_o = \frac{p_o}{k} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}}$$

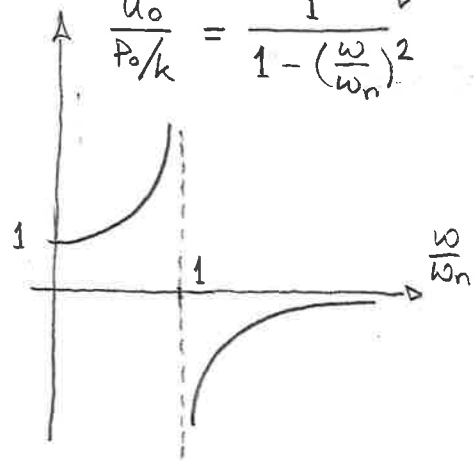
for a sinusoidal force with amplitude p_o with $\omega_n^2 = \frac{k}{m}$ the natural frequency.

The factor $\frac{p_o}{k}$ is the quasi static amplitude:

$$\omega \approx 0 \Rightarrow u_o \approx \frac{p_o}{k} = (u_{st})_o$$

Dynamic amplification

$$\frac{u_o}{p_o/k} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$



Note that if $\omega < \omega_n$ force and displ. goes in the same direction and in opposite directions if $\omega > \omega_n$

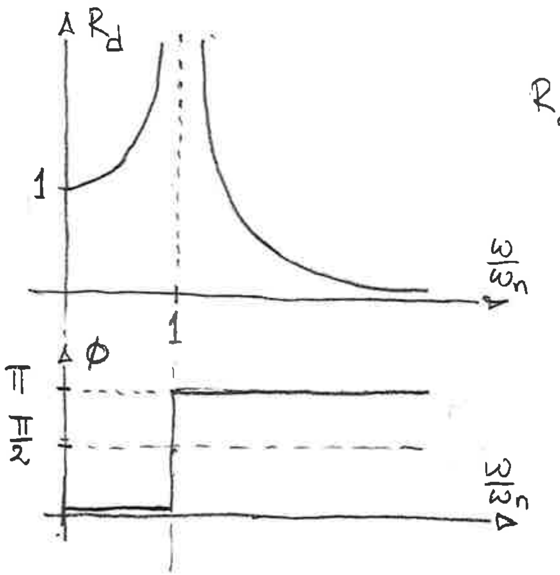
In phase and out of phase.

Dynamic resp. factor cont.

Writing the displacements as

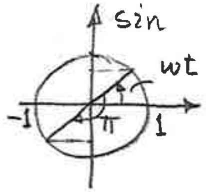
$u(t) = u_0 \sin(\omega t - \phi)$

with phase angle ϕ yields a way to define the "deformation response factor" R_d



$$R_d = \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right|$$

Unit circle:



$$\boxed{u(t) = R_d \cdot \left(\frac{P_0}{k}\right) \cdot \sin(\omega t - \phi)}$$

$\frac{P_0}{k} = (u_{st})_0$

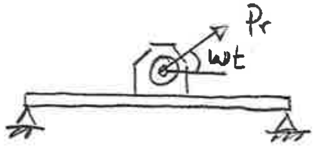
R_d shows how the quasi static amplitude is magnified.

ϕ shows if the force is in or out phase with the displacement.

Note the resonance and that the displacement amplitude goes to zero as $\omega \to \infty$.

HARMONIC FORCES: ROTATING UNBALANCE

Consider a motor on a beam (floor):



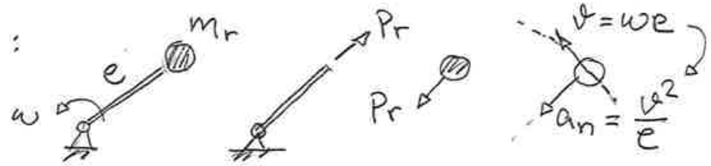
Excentricity e of rotating mass ⇒

Centroid of rotating mass is a distance e out of center of rotation.

Rotating mass m_r with angular velocity ω

Vertical force on the beam?

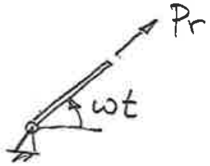
Model:



Acceleration: $a_n = e\omega^2$

NII (↙): $P_r = m_r a_n$; $P_r = m_r e \omega^2$

Vertical force p:



$p = P_r \sin \omega t$;

$$p = m_r e \omega^2 \sin \omega t$$

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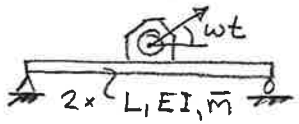
Ex

Motor resting on two floor beams



Excentricity: $e = 2\,\text{mm}$

Rot. mass: $m_r = 250\,\text{kg}$

$\omega = 17 \cdot 2\pi/\text{s}$

Motor mass: $m_m = 750\,\text{kg}$

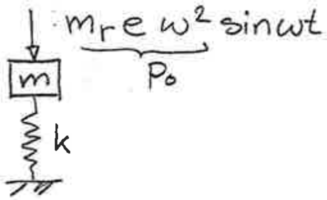
Beam mass: $\bar{m} = 16\,\text{kg/m}$

$L = 4\,\text{m}$, $EI = 1.5\,\text{MNm}^2$

Vibration ampl. at beam center?

Put half of the beam mass in the center of the beam.

Equivalent system:



Beam stiffness from table (freely supported):

$$k = 2 \cdot \frac{48EI}{L^3} = 2.25\,\text{MN/m}$$

Total mass:

$$m = 2 \cdot \frac{1}{2} \bar{m} L + m_m = 4 \cdot 16 + 750 = 814\,\text{kg}$$

Natural angular frequency $\omega_n = \sqrt{\frac{k}{m}} = 52.6\,\text{rad/s}$

Load angular frequency $\omega = 17 \cdot 2\pi = 107\,\text{rad/s}$

$$P_0 = m_r e \omega^2 = 250 \cdot 2 \cdot 10^{-3} \cdot 107^2 = 5.7\,\text{kN}$$

Ex. cont. motor on beams

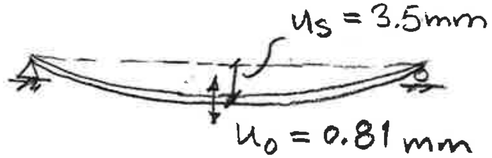
The load frequency is much higher than the natural freqveny ⇒ smal vibrations u_0

$$u_0 = \frac{P_0}{k} \left| \frac{1}{1 - (\frac{\omega}{\omega_n})^2} \right| = \frac{5700}{2.25 \cdot 10^6} \left| \frac{1}{1 - (\frac{107}{52.6})^2} \right| = 0.81\,\text{mm}$$

Static deflection: $mg = k u_s$; $u_s = \frac{mg}{k}$

$$\frac{mg}{k} = \frac{814 \cdot 9.81}{2.25 \cdot 10^6} = 3.5\,\text{mm}$$

The vibration amplitude u_0 is taking place around the static deflection:



Rem. The resonance freqvency can be passed without damage if it is done fast enough. also for lightly damped systems.

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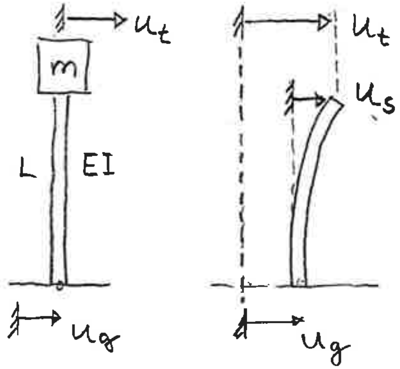
EQ. OF MOTION IN STRUCTURAL DISPL. – GROUND MOTION

Ex.



structural-displacement:

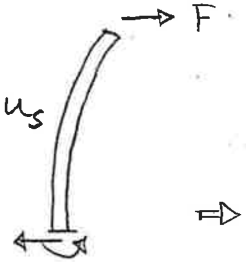
$$u_s = u_t - u_g \quad \dots (1)$$

Important quantity!

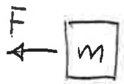
u_g : Ground displacement
u_t : Absolute —"— of mass

Beam:

$$F = \frac{3EI}{L^3} u_s$$



Mass:



$$(\rightarrow) \quad m\ddot{u}_t = -F$$

$$\Rightarrow \quad m\ddot{u}_t + \frac{3EI}{L^3} u_s = 0 \quad \dots (2)$$

(1) $\Rightarrow$ $u_t = u_s + u_g$ into (2) $\Rightarrow$

$$\boxed{m\ddot{u}_s + \frac{3EI}{L^3} u_s = -m\ddot{u}_g}$$

Equation of motion in terms of u_s

Rem. This is used in analysis of earthquake load