STRUCTURAL DYNAMIC COMPUTING

**Reduction methods for dynamic systems:**

- Static condensation
- Modal truncation
- Generalized SDOF systems
- Use of Ritz vectors

REDUCTION OF DOF'S - OVERVIEW

Computational savings can be done by the following methods:

* Static condensation

  Dof's without inertia are eliminated

* Modal truncation

  Higher modes are neglected in the modal expansion

* Generalized SDOF system

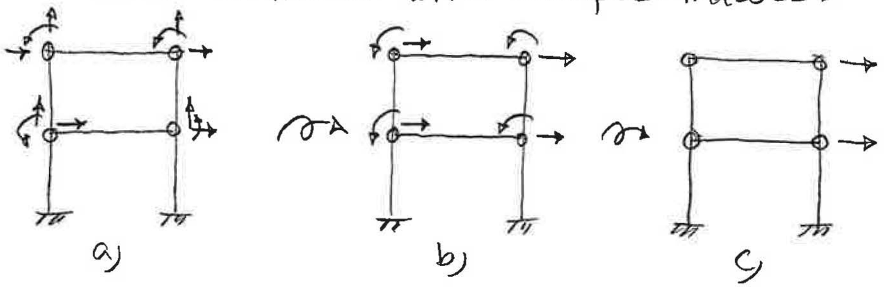
  An approximation of the first mode is used to obtain a SDOF system

* Rayleigh-Ritz method

  A special set of base vectors are used to express the displacements

# STATIC CONDENSATION OF DOFS WITHOUT INERTIA

Consider a frame with lumped masses:




a) All degrees of freedom.

b) Horizontal excitation ⇒ vertical dofs not needed in dynamic analysis

c) Beams are stiff in axial direction and zero diagonal elements in rotational dofs

Starting from a) dofs with zero inertia have displacements $u_o$

The remaining (two) dofs are dynamic dofs $u_c$.

Partitioned system:

$$\begin{bmatrix} m_{cc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u}_c \\ \ddot{u}_o \end{bmatrix} + \begin{bmatrix} k_{cc} & k_{co} \\ k_{co}^T & k_{oo} \end{bmatrix} \begin{bmatrix} u_c \\ u_o \end{bmatrix} = \begin{bmatrix} p_c \\ 0 \end{bmatrix} \quad \ldots (1)$$



Eq. (1b) can be used to eliminate $u_o$ in eq. (1a):

$$k_{co}^T u_c + k_{oo} u_o = 0 \quad ; \quad u_o = -k_{oo}^{-1} k_{co}^T u_c \quad \ldots (2)$$

Insert (2) into (1) ⇒

$$m_{cc} \ddot{u}_c + \underbrace{\left(k_{cc} + k_{co}\left(-k_{oo}^{-1} k_{co}^T\right)\right)}_{\hat{k}_{cc}} u_c = p_c(t)$$

The condensed stiffness matrix:

$$\underline{\underline{\hat{k}_{cc} = k_{cc} - k_{co} k_{oo}^{-1} k_{co}^T}}$$

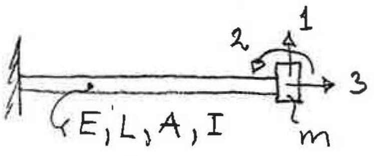
The condensed system:

$$m_{cc} \ddot{u}_c + \hat{k}_{cc} u_c = p_c(t)$$

determines the dynamic dofs

Displacements in remaining dofs are found from (2).

# EXAMPLE : STATIC CONDENSATION

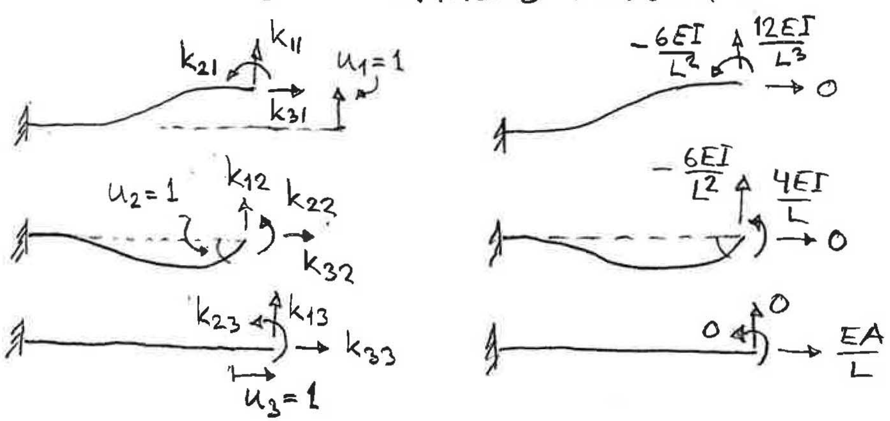



Cantilever beam : Eliminate dof's 2 and 3.

Inertia in dof 1 only*

$$\mathbb{m} = \left[\begin{array}{c|cc} m & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \mathbb{k} = \left[\begin{array}{c|cc} k_{11} & k_{12} & k_{13} \\ \hline k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{array}\right]$$

Establish the stiffness matrix :




Giving :

$$\mathbb{k} = \left[\begin{array}{c|cc} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\ \hline -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\ 0 & 0 & \frac{EA}{L} \end{array}\right]$$

Rem. *) Moment of inertia small and longitudinal stiffness is high #

EXAMPLE CONT

Partitioning :

$$\begin{bmatrix} \mathbb{m}_{tt} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} \begin{bmatrix} \ddot{u}_{tt} \\ u_o \end{bmatrix} + \begin{bmatrix} \mathbb{k}_{tt} & \mathbb{k}_{to} \\ \mathbb{k}_{ot} & \mathbb{k}_{oo} \end{bmatrix} \begin{bmatrix} u_t \\ u_o \end{bmatrix} = \begin{bmatrix} \mathbb{P}_t(t) \\ \mathbb{0} \end{bmatrix}$$

Eq. of motion :

$$\mathbb{m}_{tt} \ddot{u}_t + \hat{\mathbb{k}}_{tt} u_t = \mathbb{P}_t(t) \quad ;$$

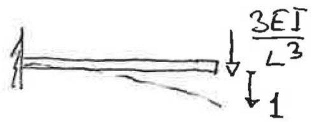
$$\hat{\mathbb{k}}_{tt} = \mathbb{k}_{tt} - \mathbb{k}_{ot}^T \, \mathbb{K}_{oo}^{-1} \, \mathbb{k}_{ot} =$$

$$= \frac{12EI}{L^3} - \begin{bmatrix} -\frac{6EI}{L^2} & 0 \end{bmatrix} \begin{bmatrix} \frac{L}{4EI} & 0 \\ 0 & \frac{L}{EA} \end{bmatrix} \begin{bmatrix} -\frac{6EI}{L^2} \\ 0 \end{bmatrix} =$$

$$= \frac{12EI}{L^3} - \begin{bmatrix} -\frac{6EI \cancel{L}}{L^2 4EI} & 0 \end{bmatrix} \begin{bmatrix} -\frac{6EI}{L^2} \\ 0 \end{bmatrix} =$$

$$= \frac{12EI}{L^3} - \frac{36EI}{4L^3} = \frac{12EI}{L^3} - \frac{9EI}{L^3} = \underline{\underline{\frac{3EI}{L^3}}}$$

Compare with cantilever 1D :




The same eq.

$$m \ddot{u} + \frac{3EI}{L^3} u = p(t)$$

#

# TRUNCATED MODAL EXPANSION

The modal expansion is truncated at $J < N$ giving an approximation :

$$u \approx \sum_{1}^{J} \phi_n q_n$$

Higher modes are left out.

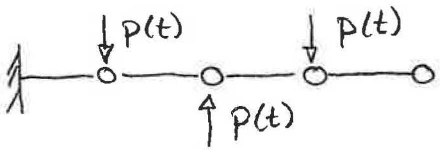
Modal coordinates are obtained for $J$ uncoupled equations.

The approximation is good if the spatial distribution of external forces triggers the lower modes:

a)

$p(t) = p_0 \sin \omega t$

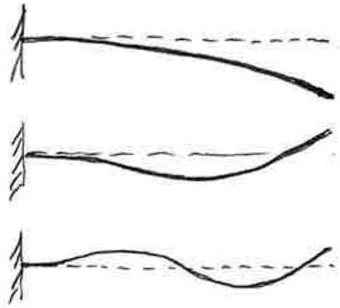
b)




- $u$ is well described by the lower modes in a)
- Higher modes have bigger influence in b)



However, the time variation of the force is also important. Consider case a) again for increasing frequeny of the load



mode 1 , $\omega_1$

mode 2 , $\omega_2$

mode 3 , $\omega_3$

Dispite the spatial distribution of the load, higer modes are triggered as the frequency increases;

$$\begin{cases} \omega \lesssim \omega_1 & \Rightarrow \text{mode 1 behavior} \\ \omega_1 < \omega \lesssim \omega_2 & \Rightarrow \text{mode 2} \quad \text{—"—} \\ \omega_2 < \omega \lesssim \omega_3 & \Rightarrow \text{—"— 3} \quad \text{—"—} \end{cases}$$

Conclusion:

The modes that are triggered depend both on the spatial distribution and on the time variation of the loading.

## OBTAINING THE REDUCED SYSTEM BY MODAL REDUCTION

After having chosen the number $J$ of the modes to keep, the modal expansion gives an approximation of the displacement vector:

$$u \approx \sum_{1}^{J} \phi_n q_n = \underbrace{\Phi_J}_{[N\times J]} \underbrace{q_J}_{[J\times 1]} \quad \dots (*)$$

The original system

$$m\ddot{u} + k u = p$$

is <u>diagonalized</u> by a transformation similar to the case of the full system:

$$\underbrace{\Phi_J^T m \Phi_J}_{M_J} \ddot{q}_J + \underbrace{\Phi_J^T k \Phi_J}_{K_J} q = \underbrace{\Phi_J^T p}_{P_J} \quad \dots (**)$$

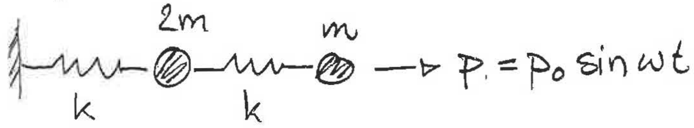
Dimensions of matrices

$$\begin{cases} M_J \text{ and } K_J : & [J\times N][N\times N][N\times J] = [J\times J] \\ q_J \text{ and } P_J : & [J\times 1] \end{cases}$$

The system (**) is solved as decoupled SDOF equations giving $q_J$. Then the physical displacement $u$ are found from (*).

## EX. MODAL TRUNCATION – TWO TO ONE DOF

Look again at the two dof system being diagonalized and solved for harmonic loading previously




$$k = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad P = P\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It was given that

$$\begin{cases} \omega_1^2 = \left(1 - \frac{1}{\sqrt{2}}\right)\frac{k}{m}, & \phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \\ \omega_2^2 = \left(1 + \frac{1}{\sqrt{2}}\right)\frac{k}{m}, & \phi_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \end{bmatrix} \end{cases}$$

Modal expansion

$$u = \Phi q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Truncate according to

$$u \approx \Phi_J q_J = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} q \quad \dots (*) \qquad (\text{one dof}; q)$$

Obtain the reduced system: $M_J \ddot{q}_J + K_J q = P_J$

$M_J = \Phi_J^T m \Phi_J$, $K_J = \Phi_J^T k \Phi_J$ and $P_J = \Phi^T p$

Ex modal trunc. cont.

$$M_J = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} m = \dots = 2m$$

$$K_J = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} k = \dots = (2-\sqrt{2})k$$

$$P_J = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} p = p$$

The reduced system

$$2m\ddot{q} + (2-\sqrt{2})k\,q = p \;;$$

$$\ddot{q} + \underbrace{\left(1-\frac{1}{\sqrt{2}}\right)\frac{k}{m}}_{\omega_1^2} q = \frac{p}{2m}$$

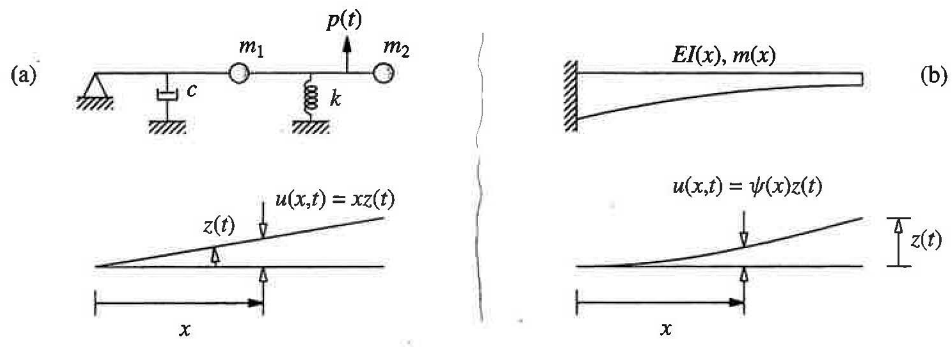
It is clear that the reduced system behaves according to the first mode

This is also seen from (*) :

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{q}{\sqrt{2}} \\ q \end{bmatrix}$$ the masses are in phase according mode 1

## GENERALIZED SDOF SYSTEMS

Systems with more than one discrete mass or even distributed mass can be treated as a SDOF system by assuming a given deflected shape $\psi(x)$




The generalized equation of motion is written as

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)$$

The generalized quantities are determined from $\psi(x)$

Useful for determining the lowest natural frequency

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}}$$

# GENERALIZED SDOF SYSTEMS

Knowledge about the system behaviour can be used to form a SDOF system from a MDOF system:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \quad \ldots (*)$$

Assume the displacements follow a particular shape $\psi$ $(N \times 1)$

$$\mathbf{u} = \psi \cdot z(t) \quad \ldots (**)$$

controlled by a single coordinate $z(t)$.

$\Rightarrow$ $\ddot{\mathbf{u}} = \psi \ddot{z}(t)$ insert into $(*)$ $\Rightarrow$

$$\mathbf{m}\psi\ddot{z} + \mathbf{k}\psi z = \mathbf{p}$$

Multiply by $\psi^T$ $\Rightarrow$

$$\underbrace{\psi^T \mathbf{m} \psi}_{m^*} \ddot{z} + \underbrace{\psi^T \mathbf{k} \psi}_{k^*} z = \underbrace{\psi^T \mathbf{p}}_{p^*}$$

The generalized SDOF-system:

$$\underline{m^* \ddot{z} + k^* z = p^*}$$

The MDOF system has been reduced to a SDOF system by the assumption of a deflected shape $\psi$ and a single coordinate $z(t)$ controling the motion.



The generalized systems initial value problem can be solved by using $(**)$

$$\mathbf{u} = \psi \cdot z \; ; \qquad \psi^T \mathbf{u} = \psi^T \psi z \; ;$$

$$z = \frac{\psi^T \mathbf{u}}{\psi^T \psi} \qquad (\psi^T \psi \text{ is a scalar})$$

The initial values for the generalized system are thus given by

$$\begin{cases} z(0) = \dfrac{\psi^T \mathbf{u}(0)}{\psi^T \psi} \\[2ex] \dot{z}(0) = \dfrac{\psi^T \dot{\mathbf{u}}(0)}{\psi^T \psi} \end{cases}$$

with $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$ being physical initial value vectors of the original mdof system.

## ESTIMATING THE LOWEST NATURAL FREQUENCY

Natural frequencies are determined from the homogeneous system:

$$(\mathbb{k} - \omega^2 \mathbb{m})\,\phi = 0$$

Assume an approximation of the first mode given by $\phi_1 \approx \psi \Rightarrow$

$(\mathbb{k} - \omega_1^2 \mathbb{m})\,\psi \approx 0$ ; $\mathbb{k}\psi \approx \omega_1^2 \mathbb{m}\psi$ ;

Multiply by $\psi^T \Rightarrow$ $\psi^T \mathbb{k} \psi \approx \omega_1^2 \psi^T \mathbb{m} \psi$ ;

$$\omega_1^2 \approx \frac{\psi^T \mathbb{k} \psi}{\psi^T \mathbb{m} \psi}$$

This is the so called <u>Rayleigh quotent</u>

<u>Rem</u>. Note the resemblance with strain energy $E_s$ and kinetic energy $E_k$

$E_s = \frac{1}{2} u^T \mathbb{k} u$ and $E_k = \frac{1}{2} \dot{u}^T \mathbb{m} \dot{u}$

$\dot{u} = \omega u$ and $E_s = E_k \Rightarrow$

$\frac{1}{2} u^T \mathbb{k} u = \omega^2 \frac{1}{2} u^T \mathbb{m} u$ ; $\omega^2 = \dfrac{u^T \mathbb{k} u}{u^T \mathbb{m} u}$

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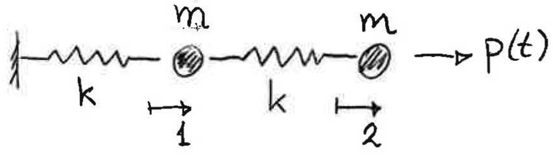
## Ex. GENERALIZED SDOF SYSTEM

Consider again the simple two dof system




$\mathbb{k} = k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad \mathbb{m} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad p = p(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Assume that the system moves in something like the first mode:

$\psi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ a linear deflection

$$u(t) = \psi\, z(t) = \begin{bmatrix} z(t) \\ 2z(t) \end{bmatrix}$$

Generalized quantities:

$$\begin{cases} k^* = [1 \;\; 2]\, k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [1 \;\; 2]\, k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overset{?}{=} 2k \\ m^* = [1 \;\; 2]\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = m\, [1 \;\; 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} \overset{?}{=} 5m \\ p^* = [1 \;\; 2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} p(t) = 2\, p(t) \end{cases}$$

Ex. cont.

The generalized SDOF system:

$$5m\,\ddot{z}(t) + 2k\,z(t) = 2\,p(t)$$

The natural frequency $\omega^2 = \dfrac{2k}{5m} = 0.4\dfrac{k}{m}$

Compare with the lowest natural frequency:

$$\omega_1^2 = 0.382\,\frac{k}{m}$$

The estimated frequency is a good approximation of the lowest natural frequency

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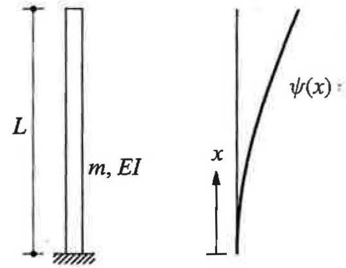
## BEAM-LIKE STRUCTURES - GENERALIZED SDOF SYSTEMS




assumed deflection

Generalized quantities

$$\left\{\begin{aligned} \tilde{m} &= \int_0^L m(x)\,[\psi(x)]^2\,dx \\ \tilde{k} &= \int_0^L EI(x)\,[\psi''(x)]^2\,dx \\ \tilde{p} &= \int_0^L p(x,t)\,\psi(x)\,dx \end{aligned}\right.$$

in the equation

$$\tilde{m}\,\ddot{z} + \tilde{k}\,z = \tilde{p}(t)$$

for motion according to the restrictions given by $\psi(x)$.

Rem. Beam properties $m$ and $EI$ can vary with $x$.

Beam like struct, cont.

Ex, <u>Lowest natural frequency estimates for a cantilever</u>

An estimate of the lowest natural frequency can be found from an assumed reasonable shape $\psi(x)$ for the lowest vibration mode.

The assumed shape must satisfy geometric boundary conditions.

**TABLE 8.5.1** NATURAL FREQUENCY ESTIMATES FOR A UNIFORM CANTILEVER

| | $\psi(x)$ | $\alpha_n$ | % Error |
|---|---|---|---|
| 1. | $3x^2/2L^2 - x^3/2L^3$ | 3.57 | 1.5 |
| 2. | $1 - \cos(\pi x/2L)$ | 3.66 | 4 |
| 3. | $x^2/L^2$ | 4.47 | 27 |

$$\omega_n = \alpha_n \sqrt{\frac{EI}{mL^4}}$$

Cantilever on previous page $\omega_n$ for 3 assumed shapes in the table

The static deflection from distributed forces 1. is a good choice of $\psi(x)$. Trigonometric functions 2. is also accurate.

*

## APPROXIMATION BY RITZ VECTORS - OVERVIEW

A reduced approximate dynamic system can be obtained by Ritz vectors.
A careful choice of a small number $J$ of vectors ($J \ll N$) yields a system with $J$ dofs, giving a good approximation to the original system with $N$ dofs.

$$u \approx \sum_{1}^{J} z_j(t)\, \Psi_j = \Psi \mathbf{z}(t), \quad [\Psi] = N \times J$$

Substituting into the standard form:

$$\mathbf{m}\Psi\ddot{\mathbf{z}} + \mathbf{c}\Psi\dot{\mathbf{z}} + \mathbf{k}\Psi\mathbf{z} = \mathbf{p}(t)$$

Premultiplying with $\Psi^T \Rightarrow$

$$\tilde{\mathbf{m}}\ddot{\mathbf{z}} + \tilde{\mathbf{c}}\dot{\mathbf{z}} + \tilde{\mathbf{k}}\mathbf{z} = \tilde{\mathbf{p}}(t)$$

The reduced system with

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi, \quad \tilde{\mathbf{c}} = \Psi^T \mathbf{c} \Psi, \quad \hat{\mathbf{k}} = \Psi^T \mathbf{k} \Psi \text{ and}$$
$$\hat{\mathbf{p}} = \Psi^T \mathbf{p}$$

has $J$ dofs as $[\mathbf{z}] = [\hat{\mathbf{p}}] = J \times 1$ and

$$[\mathbf{m}] = [\mathbf{c}] = [\mathbf{k}] = J \times J.$$

$$\boxed{\text{The usefulness of the reduced system depends entirely on good choices of Ritz vectors}}$$

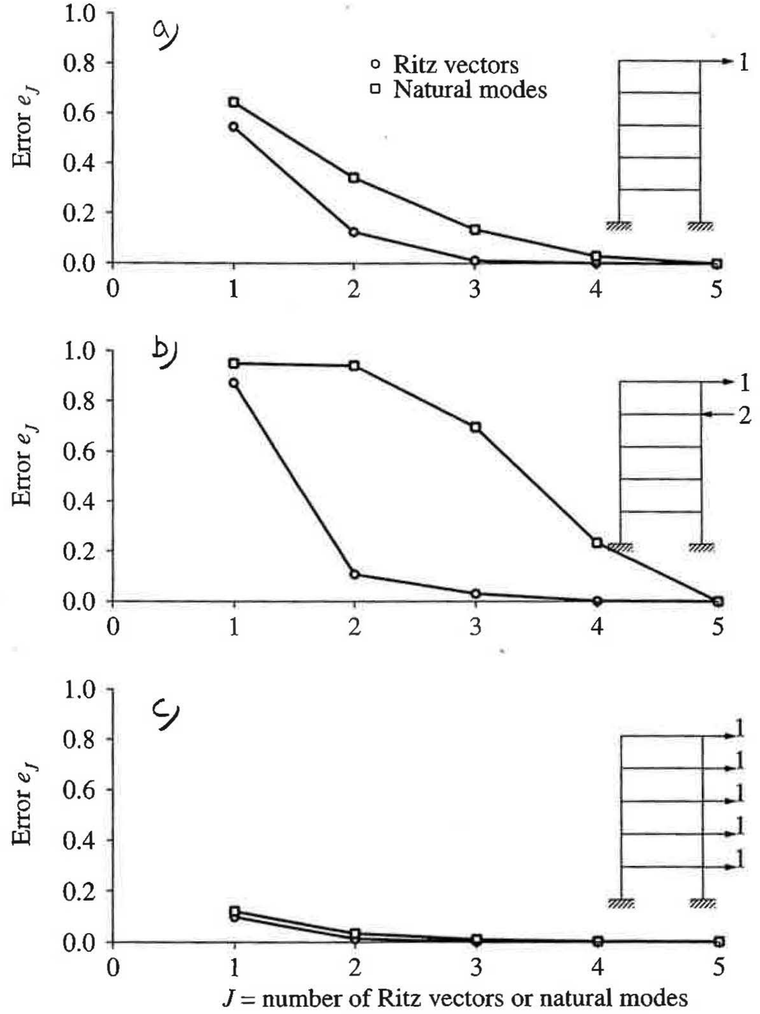
The choice of vectors can be based on:

1) Physical insight - good guess of "modes" that influence the system dynamics

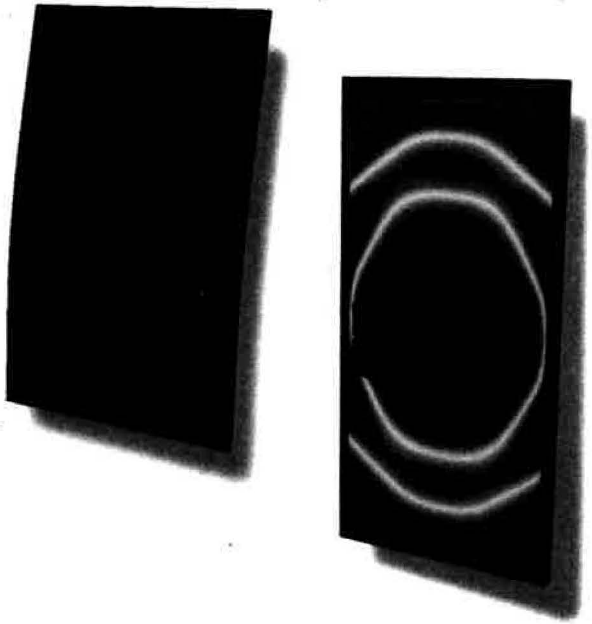
2) An automatic procedure giving Ritz vectors that conforms with the spatial distribution of the load.

Rem. In 2) Grahm-Schmidt's orthogonalization procedure is used in the so called Lanczos algorithm.

**EXAMPLE: COMPARISON OF RITZ VECTORS AND NATURAL MODES WITH DIFFERENT FORCE DISTRIBUTIONS**




case b) shows an distinct advantage of using base vectors conforming with the load distribution



# STRENGTH DESIGN METHODS FOR GLASS STRUCTURES

MARIA FRÖLING

Structural Mechanics

*Doctoral Thesis*

APPLICATION RITZ VECTORS, IMPACT ON GLASS PANES

Two shape vectors were used:

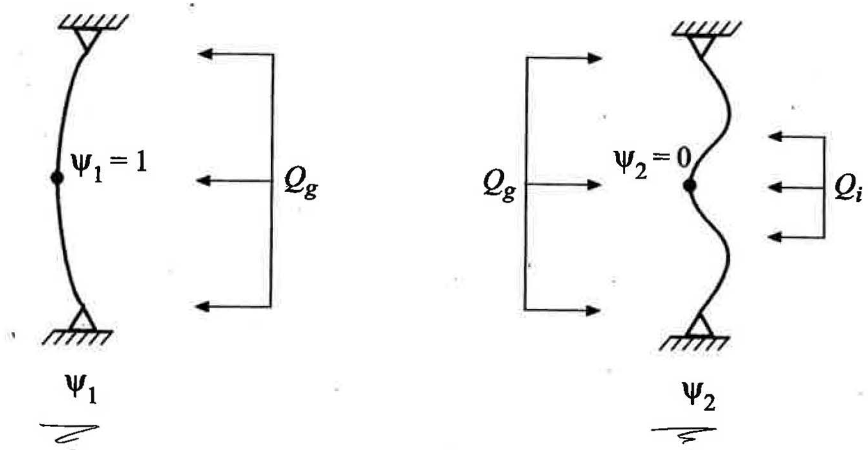



Figure 23: Construction of Ritz vectors.



M. Fröling PhD-thesis

## SOLVING THE REDUCED EQUATIONS - INITIAL VALUES

The reduced system (J dofs):

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p} \quad , \quad u = \Psi z$$

How can initial values be invoked?

$$\begin{cases} u(0) = \Psi z(0) \\ \dot{u}(0) = \Psi \dot{z}(0) \end{cases} \qquad [\Psi] = N \times J \, , \; J \ll N$$

Multiply by $\Psi^T \Rightarrow$

$$\begin{cases} \Psi^T u(0) = \Psi^T \Psi z(0) \\ \Psi^T \dot{u}(0) = \underbrace{\Psi^T \Psi}_{J \times J} \dot{z}(0) \end{cases}$$

We have two small linear systems of equations to solve in order to obtain initial values in $z$, ie. $z(0)$ and $\dot{z}(0)$.

## USING RITZ VECTORS TO OBTAIN APPROXIMATE NATURAL FREQ. & MODES

Linear combinations of Ritz vectors can also be used to approximate natural modes and frequencies by:

$$\tilde{\phi}_n = \Psi z_n \qquad [z_n] = J \times 1$$

$\tilde{\phi}_n$ is the approximate eigenvector and $z_n$ containes weight factors.

The best combination of Ritz vectors is found from the reduced eigen problem:

$$\tilde{k} z = \rho \tilde{m} z$$

with the approximate natural circular frequencies: $\omega_i^2 \approx \rho_i \qquad i = 1, \dots J$

Derivations:

Eigenvector trial solution $\tilde{\phi}$ as a linear combination of Ritz vectors

$$\tilde{\phi} = \Psi z \qquad [N \times J][J \times 1]$$

$$(k - \tilde{\omega}^2 m)\tilde{\phi} = 0 \quad ;$$

$$\Psi^T (k - \tilde{\omega}^2 m)\Psi z = 0 \quad ;$$

$$(\tilde{k} - \tilde{\omega}^2 \tilde{m}) z = 0 \qquad \ldots (*)$$

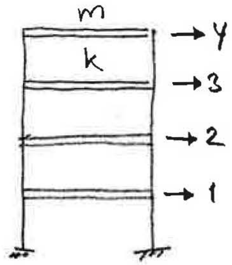
with $\tilde{k} = \Psi^T k \Psi$ and $\tilde{m} = \Psi^T m \Psi$ $[J \times J]$

Good choice of $\Psi \Rightarrow$

$$\tilde{\omega}_n \approx \omega_n \quad \text{for} \quad n = 1, 2, \ldots, J$$

The reduced system (*) also yields the weights $z_i$ giving the "best" approximation of the true eigenvectors.

# EX. FOUR STOREY FRAME – RITZ VECTORS



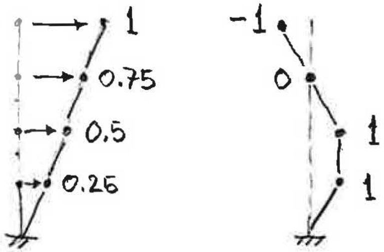

Approximate solution to eigenvalue problem:

$$(k - \omega^2 m)\phi = 0$$

$$m = m\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad k = k\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Put $k = m = 1$

Add hoc choice of Ritz vectors : $u = \Psi z$




$\Psi_1$ $\Psi_2$

$$\Psi = \begin{bmatrix} 0.25 & 1 \\ 0.5 & 1 \\ 0.75 & 0 \\ 1 & -1 \end{bmatrix}$$

Reduced system matrices :

$$\tilde{m} = \Psi^T m \Psi = \begin{bmatrix} 1.875 & -0.25 \\ -0.25 & 3.0 \end{bmatrix}$$

$$\tilde{k} = \Psi^T k \Psi = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 3.0 \end{bmatrix}$$

Solve the reduced eigen-value problem

$(\tilde{\mathbb{k}} - \tilde{\omega}^2 \tilde{\mathbb{m}}) \mathbb{z} = 0 \quad \Rightarrow$

$$\begin{cases} \tilde{\omega}_1^2 = 0.1236 \ , & \mathbb{z}_1 = \begin{bmatrix} -0.734 \\ -0.061 \end{bmatrix} \\ \omega_2^2 = 1.0 \ , & \mathbb{z}_2 = \begin{bmatrix} 0 \\ 0.577 \end{bmatrix} \end{cases}$$

"Best" approximations of $\phi_1$ and $\phi_2$ :

$$\Psi \mathbb{z}_1 = \begin{bmatrix} -0.245 \\ -0.428 \\ -0.551 \\ -0.673 \end{bmatrix} \qquad \Psi \mathbb{z}_2 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0 \\ -0.577 \end{bmatrix}$$

Exact solution:

$$\omega_1^2 = 0.1206 \qquad \omega_2^2 = 1$$

$$\phi_1 = \begin{bmatrix} -0.228 \\ -0.429 \\ -0.577 \\ -0.657 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0 \\ -0.577 \end{bmatrix}$$

Exact! $\Rightarrow$ Good choice

An extension of the gen. SDOF method.

Orthogonality:

$\phi_1^T \phi_2 = 0 \ (\approx 10^{-17})$ , $\psi_1^T \psi_2 \neq 0 \ (= -0.25)$

But $(\Psi \mathbb{z}_1)^T \Psi \mathbb{z}_2 = 0$