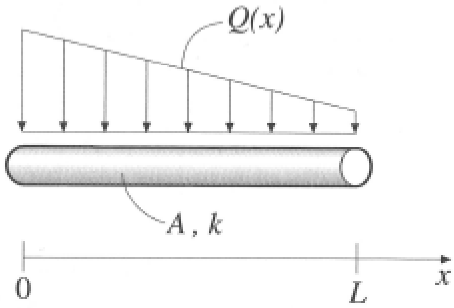
Exercises – Transient heat flow



T1.

Consider a bar of length *L*, constant cross-sectional area *A* and constant thermal conductivity *k*. The bar is made of steel with density *ρ* and heat capacity *c* and it is provided with an external heat supply $Q(x) = Q_0(6 - 6x/L)$ of dimension [W/m]. The flow of heat is assumed to be one-dimensional in the axial direction of the bar.

The balance equation for transient heat conduction is given by

$$-\frac{d}{dx}(Aq) + Q = \rho A c \frac{dT}{dt}$$

(a) Using the boundary conditions $T(0) = 80°C$ and $q(L) = 10^5$ W/m^2 for $t \geq 0$, give the strong formulation of the problem.
(b) Derive the corresponding weak formulation.
(c) Derive the FE formulation of the problem.
(d) Consider the bar as one element with linear temperature approximation and determine $\mathbf{K}$, $\mathbf{C}$, $\mathbf{f_b}$ and $\mathbf{f_l}$.
(e) Divide the bar into three equally long elements with linear temperature approximation in each of the elements and determine $\mathbf{K}$, $\mathbf{C}$, $\mathbf{f_b}$ and $\mathbf{f_l}$.

At $t = 0$, the temperature is $80°C$ in the whole bar. To determine the temperature distribution in the bar for $t > 0$, a time-stepping procedure is adopted. A linear approximation of the temperature variation between time t_i and t_{i+1} is assumed, yielding the time-stepping scheme

$$\hat{\mathbf{K}}\mathbf{a}_{\mathrm{i}+1} = \hat{\mathbf{f}}$$

where

$$\hat{\mathbf{K}} = (\mathbf{C} + \Delta t \Theta \mathbf{K})$$

$$\hat{\mathbf{f}} = \left[\left(\mathbf{C} - \Delta t \mathbf{K}(1 - \Theta)\right)\mathbf{a}_{\mathrm{i}} + \Delta t \bar{\mathbf{f}}\right]$$

The Forward Euler method is given by $\Theta = 0$ and the Backward Euler method given by $\Theta = 1$.

(f) Consider the equation system obtained in (d). For $L = 6$ m, $A = 2 \cdot 10^{-3}$ m^2, $k = 10^4$ W/mK, $Q_0 = 9$ W/m, $\rho = 7800$ kg/m^3 and $c = 490$ J/K·kg, determine $\mathbf{a}_1$ manually by calculating $\hat{\mathbf{K}}$ and $\hat{\mathbf{f}}$ with use of $\mathbf{a}_0$. Insert the boundary condition ($T(0) = 80°C$) and solve for $\mathbf{a}_1$. Employ both the Forward Euler method and Backward Euler method, using the time step $\Delta t = 120$ s, to solve the problem.

(g) With help of Matlab, plot $T(L, t)$ for $0 \leq t \leq 10$ h, using again both the Forward Euler method and Backward Euler method, with the time step $\Delta t = 120$ s, to solve the problem. What is the value at $T(L, t = 10h)$? Compare with exercise 9.1 (g).

(h) Repeat (g), now using the equation system obtained in (e).

(i) Repeat (g) and (h) using $\Delta t = 360$ s, what is observed for the different time stepping methods?

Solutions, T1

(a)

$$Ak\frac{d^2T}{dx^2} + Q = \rho Ac\frac{dT}{dt}$$

$$T(x = 0) = 80°C$$

$$q(x = L) = -\left(k\frac{dT}{dx}\right)_{x=L} = 10^5\ W/m^2$$

(b)

$$\int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}dx + \int_0^L v\rho Ac\frac{dT}{dt}dx = -(vA\cdot 10)_{x=L} + (vAq)_{x=0} + \int_0^L vQdx$$

$$T(x = 0) = 80°C$$

(c)

$$\mathbf{Ka} + \mathbf{C\dot{a}} = \mathbf{f}_\mathrm{b} + \mathbf{f}_\mathrm{l}$$

$$\mathbf{K} = \int_0^L \mathbf{B}^\mathrm{T} Ak\mathbf{B}dx\ ; \qquad \mathbf{C} = \int_0^L \mathbf{N}^\mathrm{T}\rho Ac\mathbf{N}dx$$

$$\mathbf{f}_\mathrm{b} = -[\mathbf{N}^\mathrm{T}Aq]_0^L\ ; \qquad \mathbf{f}_\mathrm{l} = \int_0^L \mathbf{N}^\mathrm{T}Qdx$$

(d)

$$\mathbf{K} = \frac{Ak}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \mathrm{W/K}; \qquad \mathbf{C} = \frac{\rho AcL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\ \mathrm{J/K}$$

$$\mathbf{f_b} = \begin{bmatrix} (Aq)_{x=0} \\ -A\cdot 10 \end{bmatrix}\ \mathrm{W}; \qquad \mathbf{f_l} = \mathrm{Q_0L}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\ \mathrm{W}$$

(e)

$$\mathbf{K} = \frac{Ak}{(L/3)}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\ \mathrm{W/K}; \qquad \mathbf{C} = \frac{\rho AcL}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\ \mathrm{J/K}$$

$$\mathbf{f_b} = \begin{bmatrix} (Aq)_{x=0} \\ 0 \\ 0 \\ -A\cdot 10 \end{bmatrix}\ \mathrm{W}; \qquad \mathbf{f_l} = \mathrm{Q_0L}\begin{bmatrix} 1.556 \\ 2.667 \\ 2.000 \\ 0.778 \end{bmatrix}\ \mathrm{W}$$

(f)

$$\mathbf{a}_0 = \begin{bmatrix} 80 \\ 80 \end{bmatrix} \quad \Rightarrow \quad \text{Forward Euler: } \widehat{\mathbf{K}} = \begin{bmatrix} 15288 & 7644 \\ 7644 & 15288 \end{bmatrix} \text{ J/K}; \quad \hat{\mathbf{f}} = \begin{bmatrix} 1860480 \\ 1830000 \end{bmatrix} \text{ J}$$

$$\text{Backward Euler: } \widehat{\mathbf{K}} = \begin{bmatrix} 15688 & 7244 \\ 7244 & 15688 \end{bmatrix} \text{ J/K}; \quad \hat{\mathbf{f}} = \begin{bmatrix} 1860480 \\ 1830000 \end{bmatrix} \text{ J}$$

Solving the resulting systems:

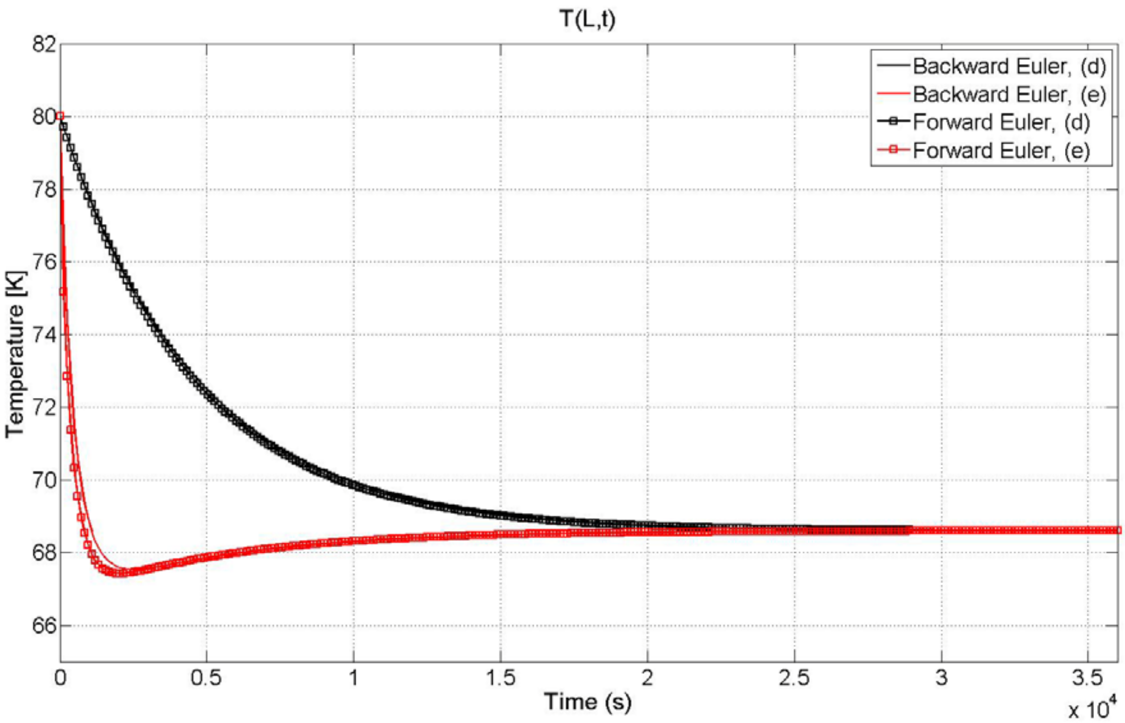
Forward Euler: $$\begin{bmatrix} 15288 & 7644 \\ 7644 & 15288 \end{bmatrix} \begin{bmatrix} 80 \\ T_{2,1} \end{bmatrix} = \begin{bmatrix} 1860480 \\ 1830000 \end{bmatrix}$$

$$\Rightarrow \quad 7644 \cdot 80 + 15288 \cdot T_{2,1} = 1830000 \quad \Rightarrow \quad \mathbf{a}_1 = \begin{bmatrix} 80 \\ 79.70 \end{bmatrix} \text{ K}$$

Backward Euler: $$\begin{bmatrix} 15688 & 7244 \\ 7244 & 15688 \end{bmatrix} \begin{bmatrix} 80 \\ T_{2,1} \end{bmatrix} = \begin{bmatrix} 1860480 \\ 1830000 \end{bmatrix}$$

$$\Rightarrow \quad 7244 \cdot 80 + 15688 \cdot T_{2,1} = 1830000 \quad \Rightarrow \quad \mathbf{a}_1 = \begin{bmatrix} 80 \\ 79.71 \end{bmatrix} \text{ K}$$

(g,h)



$T(L, t = 10h) = 68.6°C$ (in all four cases). The same value as obtained in 9.1 (g), where a steady state was assumed. Hence, at $t = 10h$ we have reached a steady state.