Non-Linear Geometry

A brief introduction
Non-linear geometry, example
Non-Linear geometry, example - kinematics

The strains may be written as:

$$\varepsilon = \frac{l_1 - l_0}{l_0}$$

The lengths of the bar in undeformed and deformed configurations: (Truncated Taylor expansion)

$$l_0 = \sqrt{b^2 + a^2} \approx b \left(1 + \frac{1}{2} \frac{a^2}{b^2}\right)$$

$$l_1 = \sqrt{b^2 + (a + u)^2} \approx b \left(1 + \frac{1}{2} \left(\frac{a + u}{b}\right)^2\right)$$

By insertion of the lengths, the strains may be written as:

$$\varepsilon = \frac{l_1 - l_0}{l_0} \approx \frac{a}{l_0} u + \frac{1}{2} \left(\frac{u}{l_0}\right)^2$$
Non-Linear geometry, example - equilibrium

Choosing a linear elastic material:  \( N = A\sigma = EA\varepsilon \)

\[
N = EA\varepsilon \approx EA \left( \frac{a u}{l_0 l_0} + \frac{1}{2} \left( \frac{u}{l_0} \right)^2 \right)
\]

Equilibrium of the central node:

\[
P = 2 N \frac{a + u}{l_1} \approx \frac{2EA}{l_0^3} \left( au + \frac{1}{2} u^2 \right) (a + u)
\]

since \( \sin\theta = (a+u)/L_1 \)

and  \( l_1 \approx b \left( 1 + \frac{1}{2} \left( \frac{a + u}{b} \right)^2 \right) \)
Non-Linear geometry, example

\[ P = 2N \frac{a + u}{l_1} \]

Tangential stiffness: \[ K_t = \frac{dP}{du} \]

Derivation of the equilibrium equation:

\[ K_t = \frac{d}{du} \left( 2N \frac{a + u}{l_0} \right) = 2 \frac{EA}{l_0} \left( \frac{a + u}{l_0} \right)^2 + 2 \frac{N}{l_0} \]

Final form of tangential stiffness:

\[ K_t = 2 \frac{EA}{l_0} \left( \frac{a}{l_0} \right)^2 + 2 \frac{EA}{l_0} \frac{2au + u^2}{l_0^2} + 2 \frac{N}{l_0} \]

\[ = K_0 + K_u + K_\sigma \]

\[ K_u = K_u(u) \]

\[ K_\sigma = K_\sigma(\sigma) \]
Non-Linear geometry, example

- First order theory: \( K_t = K_0 \)
- Second order theory: \( K_t = K_0 + K_\sigma \)
- Third order theory: \( K_t = K_0 + K_\sigma + K_u \)

\[
K_t = \frac{2EA}{l_0} \left( \frac{a}{l_0} \right)^2 + 2 \frac{EA}{l_0} \frac{2au + u^2}{l_0^2} + 2 \frac{N}{l_0} \\
= K_0 + K_u + K_\sigma
\]

\( K_u = K_u(u) \)
\( K_\sigma = K_\sigma(\sigma) \)
Non-linear geometry
- General bar element

First order:

\[ K_t = K_0 \]  
bar2e.m in Calfem

Second order:

\[ K_t = K_0 + K_\sigma \]  
bar2g.m in Calfem

Third order:

\[ K_t = K_0 + K_\sigma + K_u \]  
Not in Calfem

where

\[ \mathbf{b}_u = \begin{bmatrix} \Delta u_x (2a + \Delta u_x) & a\Delta u_y + b\Delta u_x + \Delta u_x \Delta u_y \\ a\Delta u_y + b\Delta u_x + \Delta u_x \Delta u_y & \Delta u_y (2a + \Delta u_y) \end{bmatrix} \]

and

\[ \begin{align*}
  a &= (x_2 - x_1) \\
  b &= (y_2 - y_1) \\
  \Delta u_x &= (u_3 - u_1) \\
  \Delta u_y &= (u_4 - u_2)
\end{align*} \]
Non-Linear geometry
- General solid element

The tangential element stiffness for solid elements may in most cases be written on the form:

• First order theory: \( K_t = K_0 \)
• Second order theory: \( K_t = K_0 + K_\sigma \)
• Third order theory: \( K_t = K_0 + K_\sigma + K_u \)
Solution of Non-linear Equations

Direct explicit method:

\[ \Delta u_n = K_t^{-1} \Delta P_n \]

\[ u_{n+1} = u_n + \Delta u_{n+1} \]

\( R \): residual, additive error

Divide into a number of load-steps
Out-of Balance Forces

- External forces: $P$
  
  $$P = f_b + f_l$$

- Internal forces: element forces = $I$
  
  $$I = \int_A tB^T DB dA \ a = \int_A tB^T \sigma dA$$

- Equilibrium: $P-I=0$

- In the direct explicit method: $P-I=R$

- $R$: Force Residual (Out-of-balance forces)
Newton-Raphson Method

Load steps $n = 1, 2, ...$

$P_n = P_{n-1} + \Delta P_n$

$u_n^0 = u_{n-1}$

Iterations $i = 1, 2, ...$

calculate $\varepsilon_n$ from $u_n^i$

calculate residual $R_n^i = P_{n-1} - I_n^i$

calculate $K_{t_n}^{i-1}$

$\delta u_n^i = (K_{t_n}^{i-1})^{-1} R_n^i$

$u_n^i = u_n^{i-1} + \delta u_n^i$

stop iteration when residual is ok

end of load
Stability - Linear Buckling - example

Bar with equilibrium in deformed configuration only:

**Second order theory:** \( K_t = K_0 + K_\sigma \)

\[
K_T = \frac{EA}{L} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \frac{N}{L} \begin{bmatrix}
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

**Note!** \( N = -P \) and the second term becomes negative:

\[
K_T = \frac{EA}{L} \begin{bmatrix}
\frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\
0 & -\frac{P}{L} & 0 & \frac{P}{L} \\
-\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
0 & \frac{P}{L} & 0 & -\frac{P}{L} + k_f
\end{bmatrix}
\]

\[ u_1 = u_2 = 0 \]

\[
K_T = \begin{bmatrix}
\frac{EA}{L} & 0 \\
0 & -\frac{P}{L} + k_f
\end{bmatrix}
\]

Tangent stiffness \( K_t = 0 \) when \( \det(K_t) = 0 \)

\[
\det(K_t) = 0 \quad \Rightarrow \quad P = k_f L
\]
Stability - Linear Buckling
- General problem

For a given $\sigma$ the FE-equation becomes:

$$[K_0 + K_\sigma(\sigma)] a = F$$

For a certain critical load $\lambda P = P_{cr}$ the stiffness is zero and the stability limit is reached. ($\lambda$ is a load multiplier)

$$[K_0 + K_\sigma] a = 0$$

Homogeneous equation system, non-trivial solutions exist: (eigenvalue problem)

$$(K_0 + \lambda_i K_\sigma) x_i = 0$$

$\lambda_i = the\ eigenvalues\ (force\ multipliers)$

$x_i = the\ buckling\ mode\ shapes$
Linear Buckling in ABAQUS

- Apply loads, (for example 1 N) \( \lambda_i \) will then give the buckling loads.
- Choose "Linear Perturbation" and then "Buckle" as the step. (First eigenvalue gives first buckling mode)
- Apply boundary conditions.
- Solve the eigenvalue problem.
- The solution gives the buckling modes and the force multipliers for the buckling loads.
Stability - Non-linear Buckling

• Element stiffness calculated with equilibrium in deformed configuration and updated displacement stiffness:

Third order theory: \( K_t = K_0 + K_\sigma + K_u \)

• Includes all static effects in a physical problem.
• Loading may be made until collapse is reached and post-buckling analysis may be performed.
Non-linear Buckling in ABAQUS

- Apply a load larger than the anticipated buckling load
- Choose “Static, General” problem as the step.
- Choose Nlgeom: on
- The time is fictive, dividing the load into load increments.
- Apply boundary conditions.
- A solution may not be found when a buckling load is reached.
- Preferably use displacement control.