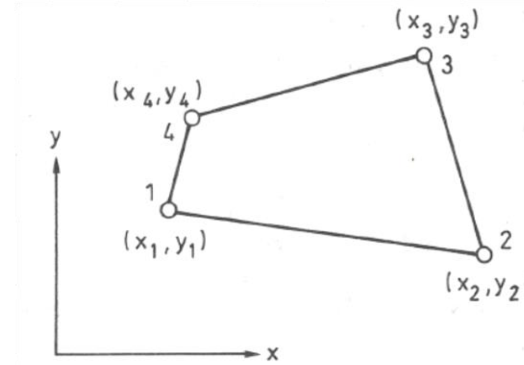
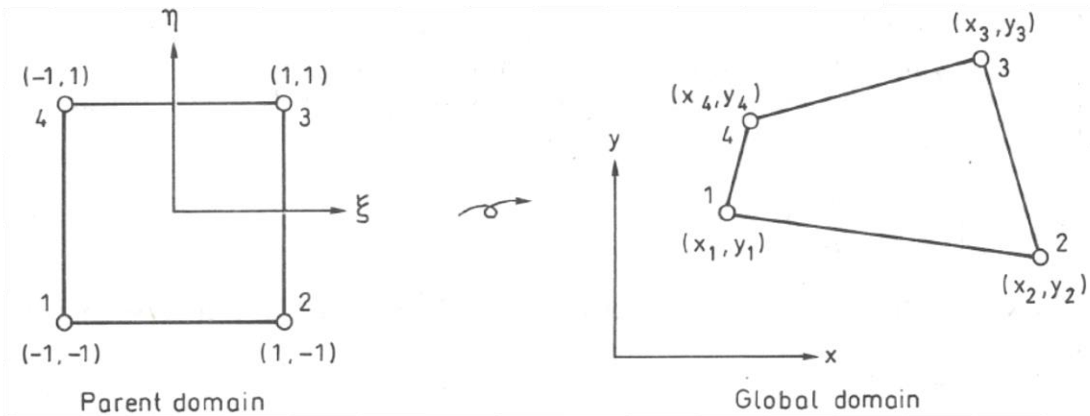
Isoparametric elements

Isoparametric elements



- Element not compatible

- Define element in a parent domain



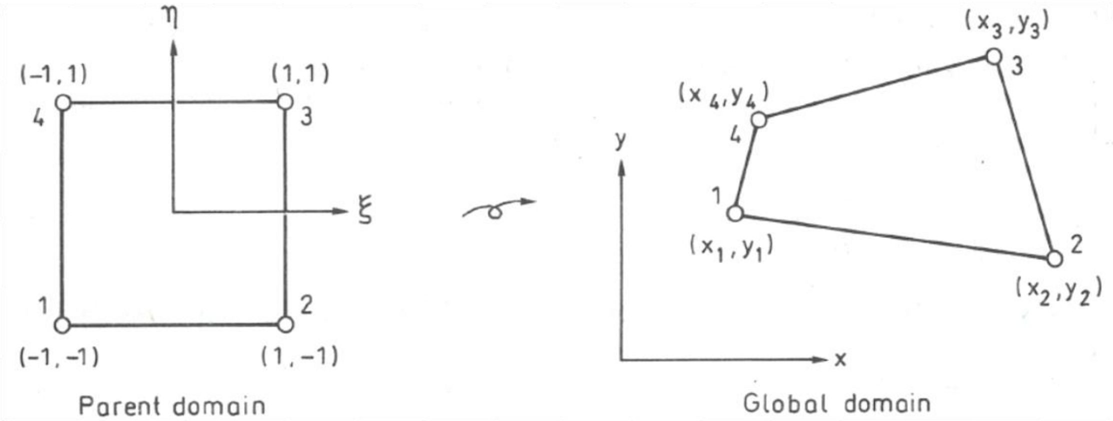
- Approximation in parent domain

$$T = T(\xi, \eta) = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$$

- Mapping to global domain

$$\boxed{x = x(\xi, \eta); \quad y = y(\xi, \eta)}$$

Mapping



- Mapping

$$\boxed{x = x(\xi, \eta); \quad y = y(\xi, \eta)}$$

- Differentiate

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}$$

- Or from global to parent domain

$$\begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$$

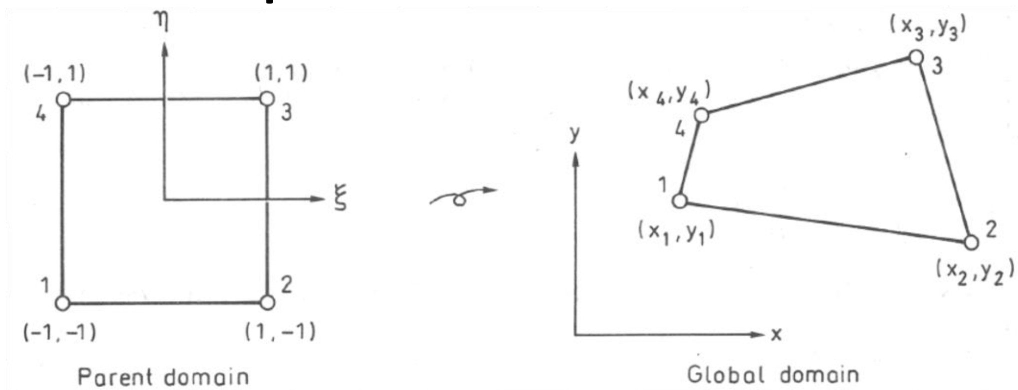
where

$$\boxed{\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}$$

- Unique mapping requirement

$$\boxed{\det \mathbf{J} > 0}$$

4-node isoparametric element



- Shape functions (from Lagrange interpolation)

$$N_1^e = N_1^e(\xi, \eta); \quad N_2^e = N_2^e(\xi, \eta); \quad N_3^e = N_3^e(\xi, \eta); \quad N_4^e = N_e(\xi, \eta)$$

$$\boxed{\begin{aligned} N_1^e &= \tfrac{1}{4}(\xi - 1)(\eta - 1); \quad N_2^e = -\tfrac{1}{4}(\xi + 1)(\eta - 1) \\ N_3^e &= \tfrac{1}{4}(\xi + 1)(\eta + 1); \quad N_4^e = -\tfrac{1}{4}(\xi - 1)(\eta + 1) \end{aligned}}$$

- Use the element shape functions for the mapping !

$$x = x(\xi, \eta) = N_1^e(\xi, \eta)x_1 + N_2^e(\xi, \eta)x_2 + N_3^e(\xi, \eta)x_3 + N_4^e(\xi, \eta)x_4$$

$$y = y(\xi, \eta) = N_1^e(\xi, \eta)y_1 + N_2^e(\xi, \eta)y_2 + N_3^e(\xi, \eta)y_3 + N_4^e(\xi, \eta)y_4$$

4-node isoparametric element

- With

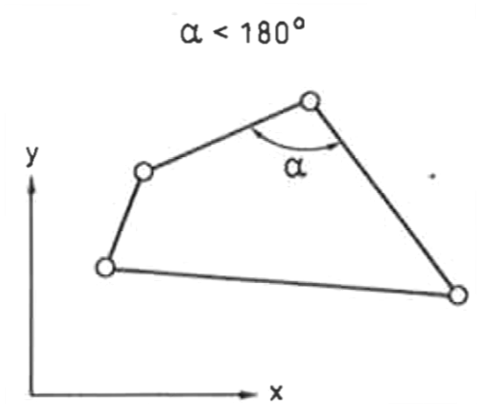
$$\mathbf{N}^e(\xi, \eta) = [N_1^e \quad N_2^e \quad N_3^e \quad N_4^e]; \quad \mathbf{x}^e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}; \quad \mathbf{y}^e = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

- The mapping can be written

$$\boxed{x = x(\xi, \eta) = \mathbf{N}^e(\xi, \eta)\mathbf{x}^e; \quad y = y(\xi, \eta) = \mathbf{N}^e(\xi, \eta)\mathbf{y}^e}$$

- Same shape functions used for approximation and mapping
- Requirements for unique mapping

$$\boxed{\det \mathbf{J} > 0}$$



4-node isoparametric element

- Mapping

$$\boxed{x = x(\xi, \eta) = \mathbf{N}^e(\xi, \eta)\mathbf{x}^e; \quad y = y(\xi, \eta) = \mathbf{N}^e(\xi, \eta)\mathbf{y}^e}$$

- Mapping of corner points:

$$x(-1, -1) = x_1; \quad x(1, -1) = x_2; \quad x(1, 1) = x_3; \quad x(-1, 1) = x_4$$

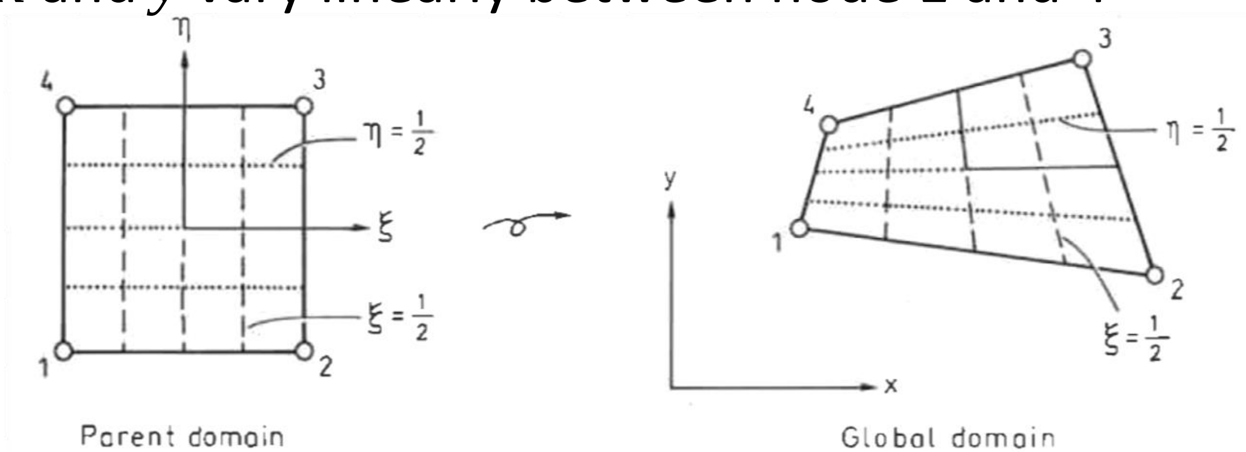
$$y(-1, -1) = y_1; \quad y(1, -1) = y_2; \quad y(1, 1) = y_3; \quad y(-1, 1) = y_4$$

- Mapping of line ξ=-1

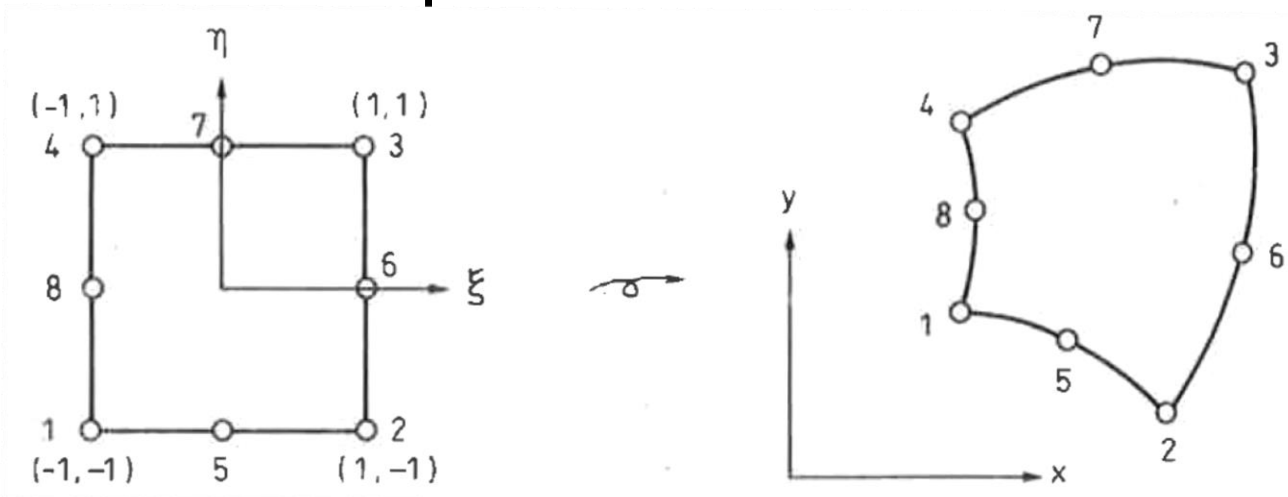
$$x(-1, \eta) = -\tfrac{1}{2}(\eta - 1)x_1 + \tfrac{1}{2}(\eta + 1)x_4$$

$$y(-1, \eta) = -\tfrac{1}{2}(\eta - 1)y_1 + \tfrac{1}{2}(\eta + 1)y_4$$

i.e. x and y vary linearly between node 1 and 4



8-node isoparametric element



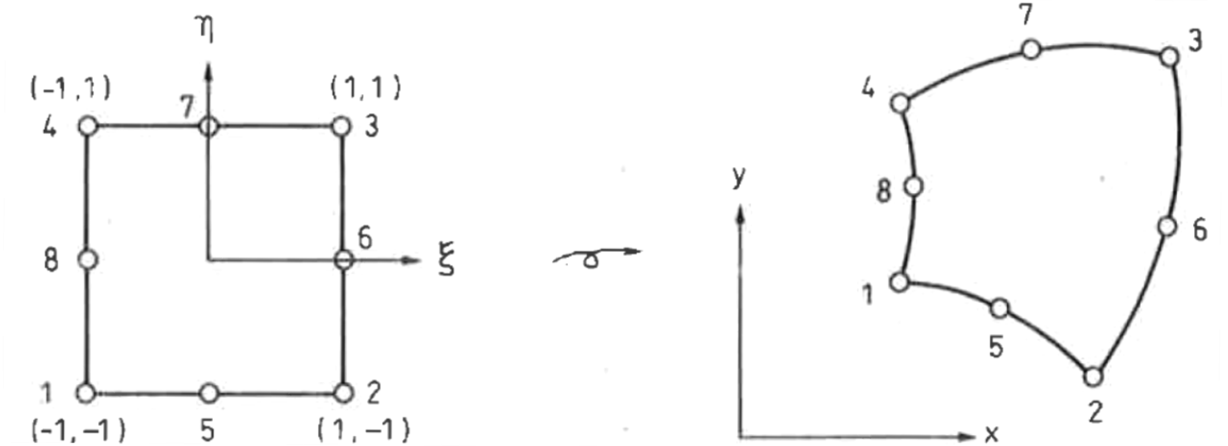
- Mapping

$$x = x(\xi, \eta) = \mathbf{N}^{e}(\xi, \eta)\mathbf{x}^{e}; \quad y = y(\xi, \eta) = \mathbf{N}^{e}(\xi, \eta)\mathbf{y}^{e}$$

- Shape functions from Lagrange interpolation polynomial

$$\begin{aligned}
N_1^{e} &= -\tfrac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta); & N_5^{e} &= \tfrac{1}{2}(1-\xi^2)(1-\eta) \\
N_2^{e} &= -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\zeta+\eta); & N_6^{e} &= \tfrac{1}{2}(1+\xi)(1-\eta^2) \\
N_3^{e} &= -\tfrac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta); & N_7^{e} &= \tfrac{1}{2}(1-\xi^2)(1+\eta) \\
N_4^{e} &= -\tfrac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta); & N_8^{e} &= \tfrac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}$$

8-node isoparametric element



- Mapping of line ξ=-1 maps to quadratic lines

$$x(-1,\eta) = -\tfrac{1}{2}(1-\eta)\eta x_1 + \tfrac{1}{2}(1+\eta)\eta x_4 + (1-\eta^2)x_8$$

$$y(-1,\eta) = -\tfrac{1}{2}(1-\eta)\eta y_1 + \tfrac{1}{2}(1+\eta)\eta y_4 + (1-\eta^2)y_8$$

- But consider nodes 1, 8 and 4 are global on a straight line

$$y_1 = \alpha x_1 + \beta; \quad y_4 = \alpha x_4 + \beta; \quad y_8 = \alpha x_8 + \beta$$

- Inserting in the second equation above

$$y(-1,\eta) = -\tfrac{1}{2}(1-\eta)\eta\alpha x_1 + \tfrac{1}{2}(1+\eta)\eta\alpha x_4 + (1-\eta^2)\alpha x_8 + \beta$$

- And using the first equation gives a linear mapping

$$y(-1,\eta) = \alpha x(-1,\eta) + \beta$$

Three-dimensional isoparametric elements

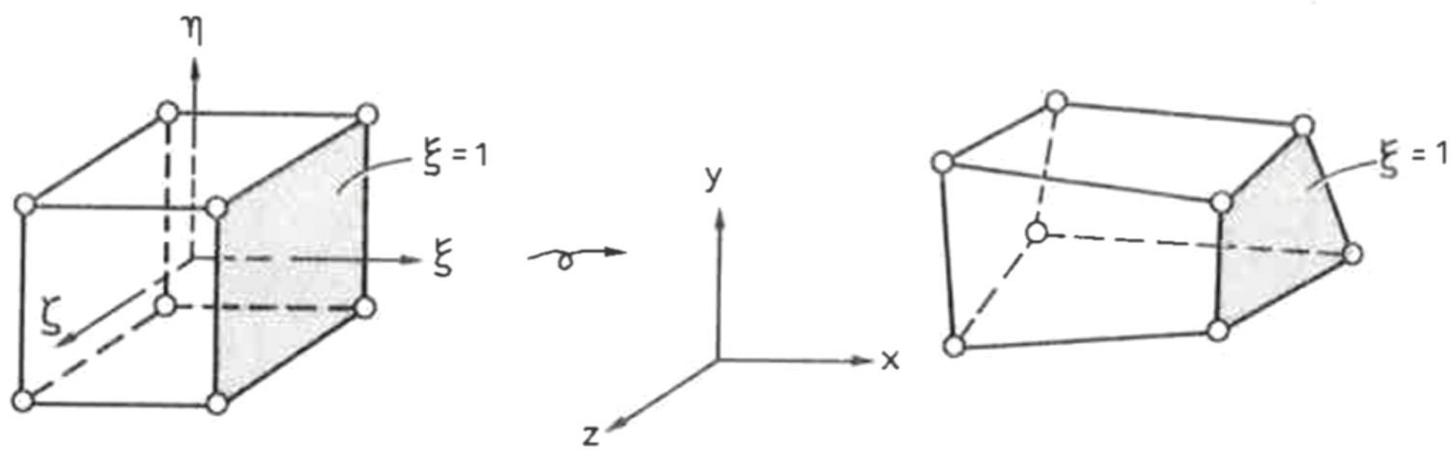


Figure 19.10 Eight-node three-dimensional isoparametric element

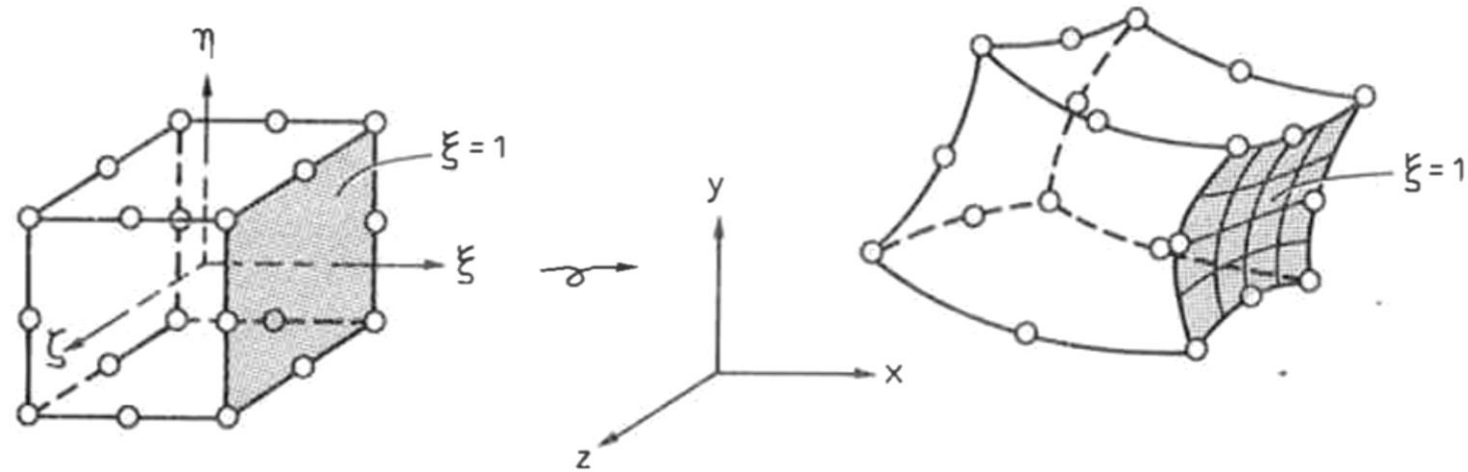


Figure 19.11 Twenty-node three-dimensional isoparametric element

Convergence criteria

- Isoparametric elements fulfil the completeness criterion if the following is satisfied

$$\boxed{\sum_{i=1}^{n} N_i^e = 1}$$

- and

> If an element behaves in a conforming, i.e. compatible, manner in the parent domain, its isoparametric version also behaves in a conforming way and no mismatch between adjacent elements exists.

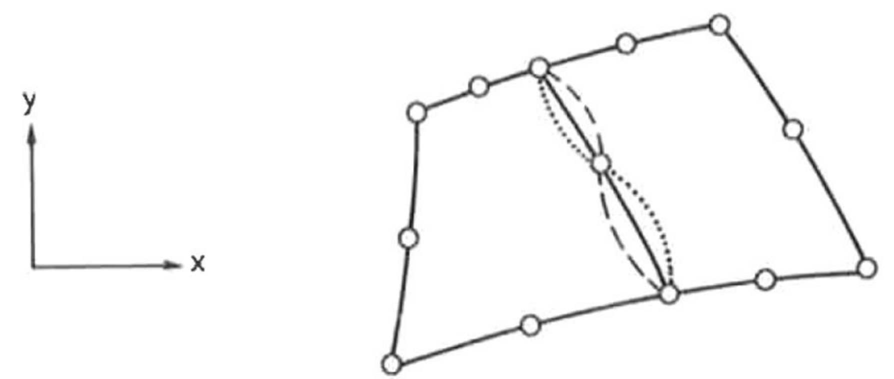


Figure 19.8 The mismatch illustrated is not possible

Finite element form of elasticity

- Two-dimensions

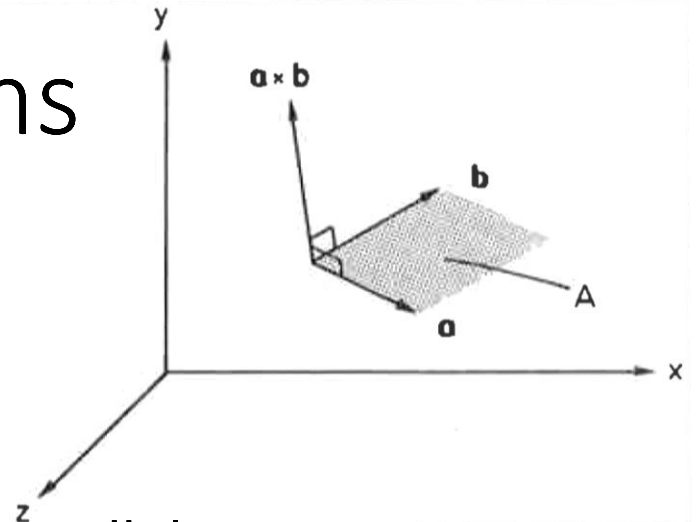
$$\left(\int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t\,\mathrm{d}A\right)\mathbf{a} = \oint_{\mathscr{L}} \mathbf{N}^{\mathrm{T}}\mathbf{t}t\,\mathrm{d}\mathscr{L} + \int_A \mathbf{N}^{\mathrm{T}}\mathbf{b}t\,\mathrm{d}A + \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}_0 t\,\mathrm{d}A$$

$$\mathbf{B}^{\mathrm{e}} = \begin{bmatrix} \frac{\partial N_1^{\mathrm{e}}}{\partial x} & 0 & \frac{\partial N_2^{\mathrm{e}}}{\partial x} & 0 & \cdots & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{\mathrm{e}}}{\partial y} & 0 & \frac{\partial N_2^{\mathrm{e}}}{\partial y} & \cdots & 0 & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \\ \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \frac{\partial N_1^{\mathrm{e}}}{\partial x} & \frac{\partial N_2^{\mathrm{e}}}{\partial y} & \frac{\partial N_2^{\mathrm{e}}}{\partial x} & \cdots & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \end{bmatrix}$$

- Three-dimensions

$$\left(\int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}V\right)\mathbf{a} = \int_S \mathbf{N}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}S + \int_V \mathbf{N}^{\mathrm{T}}\mathbf{b}\,\mathrm{d}V + \int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}_0\,\mathrm{d}V$$

Integral transformations



- Two vectors

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

- The cross product gives the area of the parallelogram

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

- If the vectors are located in the xy-plane

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ a_x b_y - a_y b_x \end{bmatrix}; \quad \text{i.e. } A = |\mathbf{a} \times \mathbf{b}| = |a_x b_y - a_y b_x|$$

- Or as

$$A = |\mathbf{a} \times \mathbf{b}| = \left| \det \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \right|$$

Integral transformations, 2-dim

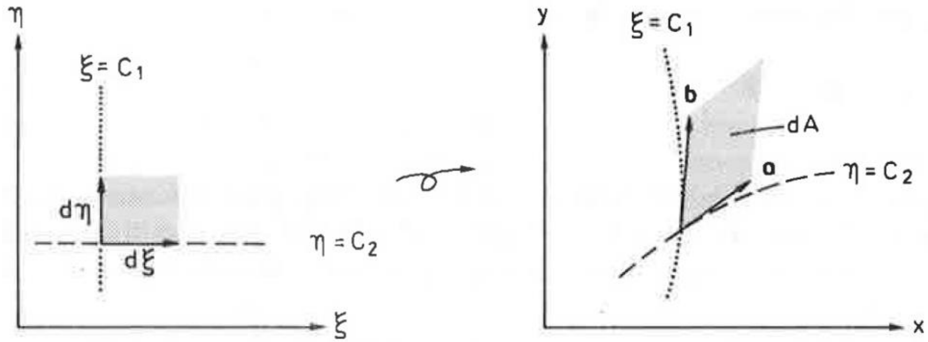
- Considering two-dimensional elements, we have $\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}$
- where

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial \mathbf{N}^e}{\partial \xi}\mathbf{x}^e; & \frac{\partial x}{\partial \eta} &= \frac{\partial \mathbf{N}^e}{\partial \eta}\mathbf{x}^e \\ \frac{\partial y}{\partial \xi} &= \frac{\partial \mathbf{N}^e}{\partial \xi}\mathbf{y}^e; & \frac{\partial y}{\partial \eta} &= \frac{\partial \mathbf{N}^e}{\partial \eta}\mathbf{y}^e \end{aligned}}$$

- Choose the vectors a and b as

$$\mathbf{a} = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \mathrm{d}\xi; \quad \mathbf{b} = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} \mathrm{d}\eta$$



- We get

$$\mathrm{d}A = \mathrm{d}\xi\,\mathrm{d}\eta \left\| \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} \right\| = \mathrm{d}\xi\,\mathrm{d}\eta \left| \det \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \right| = \mathrm{d}\xi\,\mathrm{d}\eta\,|\det \mathbf{J}|$$

$$\boxed{\mathrm{d}A = \mathrm{d}\xi\,\mathrm{d}\eta\,\det \mathbf{J}}$$

Integral transformations, 3-dim

- Generalization to three-dimensions

$$\boxed{\mathrm{d}V = \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \det \mathbf{J}}$$

- Where

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \\ \mathrm{d}\zeta \end{bmatrix} \quad \text{where} \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\mathrm{d}\eta} & \dfrac{\partial x}{\mathrm{d}\zeta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \zeta} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}$$

- This results in

$$\int_{L_\alpha} f(x)\, \mathrm{d}x = \int_{-1}^{1} f(x(\xi)) \det \mathbf{J}\, \mathrm{d}\xi$$

$$\int_{A_\alpha} f(x, y)\, \mathrm{d}A = \int_{-1}^{1} \int_{-1}^{1} f(x(\xi, \eta), y(\xi, \eta)) \det \mathbf{J}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\int_{V_\alpha} f(x, y, z)\, \mathrm{d}V = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \times \det \mathbf{J}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta$$

Transformation of boundary integrals

- In two dimensions we have

$$d\mathscr{L} = (dx^2 + dy^2)^{1/2}$$

- Either $d\eta$ or $d\xi$ is =0 along boundary, assuming $d\eta$=0

$$dx = \frac{\partial x}{\partial \xi}\, d\xi; \quad dy = \frac{\partial y}{\partial \xi}\, d\xi$$

- We then get

$$d\mathscr{L} = |d\xi| \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right]^{1/2}$$

- and the boundary integral can be written as

$$\boxed{\int_{\mathscr{L}_a} f(x, y)\, d\mathscr{L} = \int_{-1}^{1} f(x(\xi, \eta), y(\xi, \eta)) \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\delta \xi} \right)^2 \right]^{1/2} d\xi}$$

Transformation of boundary integrals

- In three dimensions the boundary is a surface where $\mathrm{d}\eta$, $\mathrm{d}\xi$ or $\mathrm{d}\zeta = 0$, assuming $\mathrm{d}\zeta = 0$

$$\mathbf{a} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \mathbf{a}_1\,\mathrm{d}\xi \quad \text{where} \quad \mathbf{a}_1 = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} \\ \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial z}{\partial \xi} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \mathbf{b}_1\,\mathrm{d}\eta \quad \text{where} \quad \mathbf{b}_1 = \begin{bmatrix} \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial z}{\partial \eta} \end{bmatrix}$$

- The infitesimal surface $\mathrm{d}S$ is then

$$\mathrm{d}S = |\mathrm{d}\xi|\,|\mathrm{d}\eta|\,|\mathbf{a}_1 \times \mathbf{b}_1|$$

- And the boundary integral is written

$$\boxed{\begin{aligned} \int_{S_a} f(x, y, z)\,\mathrm{d}S = \int_{-1}^{1}\int_{-1}^{1} & f(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)) \\ & \times |\mathbf{a}_1 \times \mathbf{b}_1|\,\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned}}$$

FE-equations in 2-dimensions

- Heat equation

$$\boxed{\mathbf{K}^e\mathbf{a}^e = \mathbf{f}_b^e + \mathbf{f}_l^e}$$

$$\mathbf{K}^e = \int_{A_\alpha} \mathbf{B}^{e\mathrm{T}}\mathbf{D}\mathbf{B}^e t \, \mathrm{d}A$$

$$\mathbf{f}_b^e = -\int_{\mathscr{L}_{h\alpha}} \mathbf{N}^{e\mathrm{T}} h t \, \mathrm{d}\mathscr{L} - \int_{\mathscr{L}_{g\alpha}} \mathbf{N}^{e\mathrm{T}} q_n t \, \mathrm{d}\mathscr{L}$$

$$\mathbf{B}^e = \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial x} \\ \dfrac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix}$$

$$\mathbf{f}_l^e = -\int_{A_\alpha} \mathbf{N}^{e\mathrm{T}} Q t \, \mathrm{d}A$$

- Gradient of shape functions

$$\begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial \xi} \\ \dfrac{\partial \mathbf{N}^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial \mathbf{N}^e}{\partial y}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial \mathbf{N}^e}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial \mathbf{N}^e}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial x} \\ \dfrac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix}$$

or

$$\begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial \xi} \\ \dfrac{\partial \mathbf{N}^e}{\partial \eta} \end{bmatrix} = \mathbf{J}^{\mathrm{T}} \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial x} \\ \dfrac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix}$$

- we get

$$\boxed{\mathbf{B}^e = \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial x} \\ \dfrac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix} = (\mathbf{J}^{\mathrm{T}})^{-1} \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial \xi} \\ \dfrac{\partial \mathbf{N}^e}{\partial y} \end{bmatrix}}$$

$$\boxed{\mathbf{K}^e = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} \dfrac{\partial \mathbf{N}^{e\mathrm{T}}}{\partial \xi} & \dfrac{\partial \mathbf{N}^{e\mathrm{T}}}{\partial \eta} \end{bmatrix} \mathbf{J}^{-1}\mathbf{D}(\mathbf{J}^{\mathrm{T}})^{-1} \begin{bmatrix} \dfrac{\partial \mathbf{N}^e}{\partial \xi} \\ \dfrac{\partial \mathbf{N}^e}{\partial \eta} \end{bmatrix} t \det \mathbf{J} \, \mathrm{d}\xi \, \mathrm{d}\eta}$$

Elasticity finite element equations

- Two dimensions

$$\left(\int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t\,\mathrm{d}A\right)\mathbf{a} = \oint_{\mathscr{L}} \mathbf{N}^{\mathrm{T}}\mathbf{t}t\,\mathrm{d}\mathscr{L} + \int_A \mathbf{N}^{\mathrm{T}}\mathbf{b}t\,\mathrm{d}A + \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}_0 t\,\mathrm{d}A$$

$$\mathbf{B}^{\mathrm{e}} = \begin{bmatrix} \frac{\partial N_1^{\mathrm{e}}}{\partial x} & 0 & \frac{\partial N_2^{\mathrm{e}}}{\partial x} & 0 & \cdots & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{\mathrm{e}}}{\partial y} & 0 & \frac{\partial N_2^{\mathrm{e}}}{\partial y} & \cdots & 0 & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} \\ \frac{\partial N_1^{\mathrm{e}}}{\partial y} & \frac{\partial N_1^{\mathrm{e}}}{\partial x} & \frac{\partial N_2^{\mathrm{e}}}{\partial y} & \frac{\partial N_2^{\mathrm{e}}}{\partial x} & \cdots & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial y} & \frac{\partial N_{n_{\mathrm{e}}}^{\mathrm{e}}}{\partial x} \end{bmatrix}$$

- Use that

$$\boxed{\begin{bmatrix} \frac{\partial N_i^{\mathrm{e}}}{\partial x} \\ \frac{\partial N_i^{\mathrm{e}}}{\partial y} \end{bmatrix} = (\mathbf{J}^{\mathrm{T}})^{-1} \begin{bmatrix} \frac{\partial N_i^{\mathrm{e}}}{\partial \xi} \\ \frac{\partial N_i^{\mathrm{e}}}{\partial \eta} \end{bmatrix}}$$