AN EQUVALENCE BETWEEN NONLOCAL PLASTICITY AND GRADIENT PLASTICITY

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Detta är en tom sida!
An equivalence between nonlocal plasticity
and gradient plasticity

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Abstract The paper addresses a certain nonlocal plasticity model, where the consistency condition may be rewritten from a Fredholm equation of the second kind into a second order differential equation. Hereby, a case of equivalence between a nonlocal model and a gradient model will be found.

1 Introduction

In strain softening, the yield limit decreases when plastic deformation localizes into shear bands and fracture zones. The mechanism behind plastic deformation, is movement of dislocations on the microscale of the material. Classical continuum plasticity theories are not able to describe softening after onset of localization and the inability arises from the fact that such theories contain no information about the size of the localization zone; a length scale is absent. In the nonlocal theory adopted here, a length enters as a material parameter by allowing a dependency on so called nonlocal variables. A nonlocal variable is a weighted average of the local variable over all the material points in the body and the length parameter determines how the value of the variable at a certain point are weighted. The length scale should reflect the ability of the microstructure to submit information to neighboring points within a certain distance. Being a remainder of such quantities as the distance between dislocations and grains, the length is a material property linking the microstructure to the continuum.


The word nonlocal is occasionally used as a collecting name on higher-order theories including both gradient theories and material descriptions containing body integral expressions. To distinguish, gradient theories are known as weakly nonlocal, and integral expressions as strongly nonlocal. Nonlocal theory will in this text refer to the latter. Gradient models of the type considered here are found in Bazant and Belytschko (1985), Mühlhaus and Aifantis (1991) and deBorst and Mühlhaus (1992).
In the present paper, a nonlocal model with mixed local hardening and non-local softening, developed in Strömberg and Ristinmaa (1996), shall be adopted. The model is based on the assumption that hardening is caused by movement of dislocations and when hardening occurs at one point, the surrounding points will be softer. It will be shown that the nonlocal model is equivalent to a gradient model for a one-dimensional formulation, and a certain weighting function.

2 Nonlocal plasticity model

A nonlocal field $\{\kappa\}$ is related to the local field $\kappa$ by the definition

$$\{\kappa\}(x) = \frac{1}{V_r(x)} \int_B \alpha(x, \xi) \kappa(\xi) d\xi$$

(1)

where the integration ranges over the entire body $B$, $\alpha$ is an averaging function and $V_r(x)$ is the so-called representative volume defined by

$$V_r(x) = \int_B \alpha(x, \xi) d\xi$$

(2)

This definition implies that for a constant local field $\kappa$, fulfils $\{\kappa\} = \kappa$. As averaging functions $\alpha$, are considered positive, symmetric functions that have maximum at one point $\xi = x$ and depend on a length parameter $l$. The averaging function will be defined for arbitrary small $l$. When $l$ approaches zero, $\{\kappa\} = \kappa$, that is the operator $\{}$ becomes the identity and we obtain the local field. This corresponds to choosing $\alpha = \delta$, $\delta$ being Dirac’s delta function. Two functions that could be considered are the Gaussian and the linear exponential; $e^{-\pi(x-\xi)^2/l^2}$, $e^{-2|x-\xi|/l}$. The Gaussian was used by Bazant and Lin (1988), and Strömberg and Ristinmaa (1996). Another possibility is to take a restricted cosine function

$$\alpha(x, \xi) = \begin{cases} \frac{1}{2a} \left( \cos \frac{\pi(x-\xi)}{l} + 1 \right) & \text{when } \xi \in [x - al, x + al] \\ 0 & \text{else} \end{cases}$$

(3)

where $a$ is a positive parameter that determines how far away we wish to consider an influence of other points. We may note that the cosine have more than one local maximum when $a > 2$. Functions of this kind are found in the literature. For instance, a symmetric $\alpha$ with a local minimum between two maxima was used by Bazant (1994), to describe the influence of microcracks in localization analysis.

Above was considered an isotropic volume averaging. For materials that have internal selected directions due to orientation of microstructure, other formulations could be suggested. For example, in Planas et. al (1994), the Rankine
yield criterion together with averaging in the direction of largest principal stress is considered. To model microcracks and damage, Bazant (1994), uses a form of directional averaging. The restricted cosine (3) was used in Strömberg (1995), to obtain closed form solutions.

In the present paper we shall consider the onedimensional case with the averaging function

\[
\alpha(x, \xi) = \begin{cases} 
A(nl - x)(nl + \xi) & -nl \leq \xi \leq x \leq nl \\
A(nl - \xi)(nl + x) & -nl \leq x \leq \xi \leq nl
\end{cases}
\]  

(4)

where \( n \) is an integer and \( A \) is to be determined from a normalization. This type of function are found in the literature of Fredholm integral equations since it makes it possible to rewrite a homogenous Fredholm integral equations into a second order differential equation with boundary conditions, cf. Tricomi (1957). This fact will be demonstrated in the next section. The function \( \alpha(x, x) \), the so-called envelope is seen in Figure 1, together with the functions \( \alpha(x, \xi_1) \) and \( \alpha(x, \xi_2) \) for \( \xi_1 = -nl/2 \) and \( \xi_2 = -nl/4 \).

![Figure 1: The averaging function \( \alpha(x, x) \), \( \alpha(x, \xi_1) \) and \( \alpha(x, \xi_2) \) for \( \xi_1 = -nl/2 \) and \( \xi_2 = -nl/4 \).](image)

The averaging functions are normalized so that the representative volume \( V_r \) given by (2) will be

\[
V_r = l, \ l^2, \ l^3
\]

(5)
at the centroids of the infinite bar, square and cube, respectively. This differs from the normalization used by Planas et al. (1994), where it is assumed that \( \alpha(0) = 1 \) and the representative volume is omitted. From the normalization (5) we achieve that the factor in (4) is a constant given by

\[
A = \frac{1}{n^3 l^2}
\]

if not too close to the boundaries of the body.

2.1 Governing equations

When body forces are absent, the static equilibrium equations are

\[
div \sigma = 0
\]

\( \sigma \) denotes the stress tensor with components \( \sigma_{ij} \). The scalar product between tensors, a contraction in two indices, is written \( \sigma : \sigma = \sigma_{ij}\sigma_{ij} \) with the summation convention. We will consider isotropic bodies and the constitutive equation relating stress to strain reads

\[
\sigma = D_e : (\varepsilon - \varepsilon_p) = 2G(\varepsilon - \varepsilon_p) + 2G \frac{\nu}{1-2\nu} I tr(\varepsilon - \varepsilon_p)
\]

where \( \varepsilon \) is the total strain tensor and \( \varepsilon_p \) is the plastic strain tensor. \( G \) is the shear modulus and \( \nu \) is Poisson’s ratio and the elastic modulus \( E = 2G(1 + \nu) \).

For the trace of a tensor we write \( tr\sigma = \sigma_{kk} \). The deviatoric part \( \sigma' \) of a tensor \( \sigma \) is given by

\[
\sigma' = \sigma - \frac{1}{3} I tr\sigma
\]

where \( I \) is the second order identity tensor. By \( |\sigma'| \) is meant the scalar length \( \sqrt{\sigma' : \sigma'} \).

The nonlocal plasticity model considered here is a von Mises model with generalized isotropic hardening, defined as a mixed local/nonlocal hardening via the parameters \( h \) and \( H \). The yield function is given by

\[
f(\sigma, \kappa, \{\kappa\}) = \sigma_e - h\kappa - H\{\kappa\} - \sigma_y
\]

where \( \kappa \) is the hardening/softening field that determines the alteration in yield stress, \( \sigma_y \) is the initial yield stress and the effective stress is defined by

\[
\sigma_e = \left( \frac{3}{2} \right)^{1/2} |\sigma'|
\]
A model of this kind, with $h$ interchanged by $(1 - m)h$ and $H$ by $mh$, $m$ being a parameter, was proposed by Strömberg and Ristinmaa (1996). The evolution laws for the internal variables are assumed to be

$$\dot{e}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma} = \frac{3}{2} \frac{\sigma'}{\sigma_e} \dot{\lambda}$$

$$\kappa = \dot{\lambda}$$

The plastic multiplier $\dot{\lambda}$ is calculated from the consistency condition, $\dot{f} = 0$, which due to (10) reads

$$\frac{3}{2} \frac{\sigma'}{\sigma_e} : \dot{\sigma} - h\dot{\lambda} - H \{\dot{\lambda}\} = 0$$

for points that satisfy $\dot{\lambda} > 0$. We note that (14) is a Fredholm equation of the second kind, cf. Appendix A.

## 3 Gradient model

We will assume the same governing equations as above except for the yield function. For the gradient model we take

$$f(\sigma, \kappa) = \sqrt{\frac{3}{2} |\sigma'|} - h\kappa + c\nabla^2 \kappa + d\nabla^4 \kappa - \sigma_y$$

with the constant material parameters $h$, $c$ and $d$. This is similar to the one considered by Muhlhaus and Aifantis (1991). For $d = 0$, the model was solved by Bazant and Belytschko (1985), de Borst and Muhlhaus (1994) in the one-dimensional case. The consistency condition, $\dot{f} = 0$, in this case reads

$$\frac{3}{2} \frac{\sigma'}{\sigma_e} : \dot{\sigma} - h\dot{\lambda} + c\nabla^2 \dot{\lambda} + d\nabla^4 \dot{\lambda} = 0$$

for points that satisfy $\dot{\lambda} > 0$. We note that (16) is a fourth order differential equation and the solution requires four boundary conditions.

## 4 Equivalence for a onedimensional problem

For a one-dimensional bar problem, the consistency condition in the nonlocal model proposed here, will be an integral equation that possesses a closed form solution. Moreover, it will be equivalent to the consistency condition of the gradient model with $d = 0$ and boundary conditions.
The body considered is a bar with length $L$, seen below. The origin is in the middle and the $x$-coordinate along the axis. The bar is loaded in displacement control with zero displacement at $x = -L/2$ and prescribed displacement $u$ at $x = L/2$.

The yield stress is assumed to be less than $\sigma_y$ in the region $x \in [-nl, nl]$, with $n$ and $l$ from (4), and thus plasticity will be confined to this zone. As was stated (8) and (12), the constitutive equations are

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\lambda})$$  \hspace{1cm} (17)

at points where plasticity occurs and

$$\dot{\sigma} = E\dot{\varepsilon}$$  \hspace{1cm} (18)

outside. We will assume that body forces are absent and equilibrium then gives

$$\frac{d\sigma}{dx} = 0 \ , \ \frac{d\dot{\sigma}}{dx} = 0$$  \hspace{1cm} (19)

Hence $\sigma$ and $\dot{\sigma}$ are independent of $x$.

4.1 Gradient model

Here we shall adopt the gradient model (15) with $d = 0$. The plastic zone is $x \in [-nl, nl]$ from the preliminaries and we have the consistency condition for determination of $\dot{\lambda}$ that due to (16) reads

$$\frac{d^2\dot{\lambda}}{dx^2} - \frac{h}{c} \dot{\lambda} = -\frac{1}{c} \dot{\sigma}$$  \hspace{1cm} (20)

It is assumed that $(-h/c)$ is positive and then the term $(c/h)^{1/2}$ has the dimension of length. To solve the differential equation above, we need two boundary conditions and these are provided by the symmetry condition

$$\dot{\lambda}(x) = \dot{\lambda}(-x)$$  \hspace{1cm} (21)

and that no plastic strains occurs at the boundaries of the zone

$$\dot{\lambda}(-nl) = \dot{\lambda}(nl) = 0$$  \hspace{1cm} (22)

The last condition implies that the strain distribution is a continuous function, or putting it differently, if we require that there should be no jump in the strain.
rate field we must fulfil (22). The solution is the sum of the solution to the homogenous equation and a particular solution

$$\dot{\lambda} = \dot{\lambda}_h + \dot{\lambda}_p$$  \hfill (23)

where \( \dot{\lambda}_h \) fulfils

$$\frac{d^2 \dot{\lambda}_h}{dx^2} - \frac{h}{c} \dot{\lambda}_h = 0$$  \hfill (24)

and \( \dot{\lambda}_p \) is given by

$$\dot{\lambda}_p = \frac{\dot{\sigma}}{h}$$  \hfill (25)

### 4.2 Nonlocal model

In the plastic zone \( x \in [-nl, nl] \), the consistency condition (14) is a Fredholm equation of the second kind

$$\dot{\lambda} = -\frac{H}{h} \{ \dot{\lambda} \} + \frac{\dot{\sigma}}{h}$$  \hfill (26)

and the averaging function is given by (4) and (6). For the solution, we shall use the Hilbert-Schmidt theory from Appendix A. If \( -\frac{H}{h} \) is an eigenvalue of \( \alpha \) and if \( \frac{\dot{\lambda}}{h} \) is orthogonal to the corresponding eigenfunction, the solution will be of the kind (39). Here, we shall assume that the last factor in (39) is not present in the solution. This turns out to be equivalent to the requirement that \( \dot{\lambda} \) should be zero only at the boundaries of the plastic zone and nonzero within, i.e \( \dot{\lambda}(x) > 0 \; \; x \in [-nl, nl] \). With these preliminaries, we achieve that the solution could be decomposed as in (23)

$$\dot{\lambda} = \dot{\lambda}_h + \dot{\lambda}_p$$  \hfill (27)

with \( \dot{\lambda}_h \) fulfilling the homogeneous equation

$$\dot{\lambda}_h = -\frac{H}{h} \{ \dot{\lambda}_h \}$$  \hfill (28)

and

$$\dot{\lambda}_p = \frac{\dot{\sigma}}{h}$$  \hfill (29)

The particular solution (29) is the same as (25). Next, we shall show that the homogeneous equation is the same as (24) with a proper choice of the material parameters. A differentiation of (28) gives

$$\frac{d^2 \dot{\lambda}_h}{dx^2} = -\frac{H}{h} \frac{d^2}{dx^2} \{ \dot{\lambda}_h \}$$  \hfill (30)
Performing the differentiation of the right hand side gives with (4) and (6)

\[
\frac{d^2}{dx^2} \{ \dot{\lambda}_h \} = -\frac{1}{l} \frac{1}{n^3 l^2} 2nl \dot{\lambda}_h = -\frac{2}{n^2 l^2} \dot{\lambda}_h
\]  

(31)

Upon substitution in (30)

\[
\frac{d^2 \dot{\lambda}_h}{dx^2} = \frac{H}{h} \frac{2}{n^2 l^2} \dot{\lambda}_h
\]  

(32)

An inspection of (24) gives by choosing

\[
\frac{H}{h} \frac{2}{n^2 l^2} = \frac{h}{c}
\]  

(33)

we arrive at the same differential equation as for the gradient model. If the preliminaries for the decomposition (27) should be fulfilled, we must have that \(-\frac{H}{h}\) is the eigenvalue to the eigenfunction that is zero only at the boundaries. This could be determined by the Fredholm theory, however the easiest way is to solve the differential equation (32) and impose the boundary conditions (22) to the total solution (28).

5 Conclusions

A nonlocal model related to the model proposed by Strömberg and Ristinmaa (1996), was developed. It was found that the consistency condition could be rewritten from a Fredholm equation of the second kind into a second order differential equation. Hereby, a case of equivalence between a nonlocal model and a gradient model was found.

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References


A Fredholm integral equations

These results for integral equations are taken from Hildebrand (1965), Arfken (1950).

Let \( a(x) \), \( F(x) \) and \( K(x, \xi) \) be known functions and \( a \) and \( \mu \) known scalar parameter. The equation

\[
ay(x) = F(x) + \mu \int_{B} K(x, \xi)y(\xi)d\xi
\]  

(34)

is called a Fredholm equation for the unknown function \( y(x) \). \( K(x, \xi) \) is called the kernel.

When \( a \equiv 0 \) (34) is a Fredholm equation of the first kind and when \( a \equiv 1 \), (34) is a Fredholm equation of the second kind. For \( F = 0, a \equiv 1 \) the equation is said to be homogeneous.

The kernel is called symmetric if \( K(x, \xi) = K(\xi, x) \). For symmetric kernels certain properties are collected in the Hilbert-Schmidt theory. The vocabulary and results are similar to those in the theory of symmetric matrices. Functions \( \varphi_j \) that fulfill the homogeneous equation

\[
\varphi_j(x) = \mu_j \int_{B} K(x, \xi)\varphi_j(\xi)d\xi \quad \text{no sum over } j
\]  

(35)

are called eigenfunctions and \( \mu_j \) are eigenvalues to \( K \) in \( B \). The kernel generates its eigenfunctions according to

\[
\int_{B} K(x, \xi)h(\xi)d\xi = \sum_{j=0}^{n} a_j \varphi_j(x)
\]  

(36)

where \( h(\xi) \) is an arbitrary function, \( a_j \) are scalar quantities and \( n \) is an integer that is equal or less than the number of linearly independent eigenfunctions. The kernel may be expanded in the eigenfunctions

\[
K(x, \xi) = \sum_{j=0}^{n} \frac{1}{\mu_j} \varphi_j(x)\varphi_j(\xi)
\]  

(37)

Here, \( n \) is the number of linearly independent eigenfunctions. Next, some results concerning the solutions. From (35) and (34), we see that the homogeneous equation \( (F(x) = 0) \) has no solution unless \( \mu \) is an eigenvalue. The solution to a Fredholm equation of the second kind, that is (34) with \( a \equiv 1 \), is

\[
y(x) = F(x) + \mu \sum_{j=0}^{\infty} \int_{B} F(\xi)\varphi_j(\xi)d\xi \frac{1}{\mu_j - \mu} \varphi_j(x)
\]  

(38)
if $\mu$ is not an eigenvalue. If $\mu$ is an eigenvalue $\mu_h$, the solution will be

$$y(x) = F(x) + a_h \varphi_h(x) + \mu_h \sum_{j=0(j\neq h)}^{\infty} \frac{\int_B F(\xi) \varphi_j(\xi) d\xi}{\mu_j - \mu_h} \varphi_j(x)$$  \hspace{1cm} (39)$$

where $a_h$ is an undetermined constant and $F$ has to be orthogonal to the eigenfunction $\varphi_h(x)$, i.e.

$$\int_B F(\xi) \varphi_h(\xi) d\xi = 0$$  \hspace{1cm} (40)$$

Hence in this case the solution may not exist and if it exists it is not fully determined.