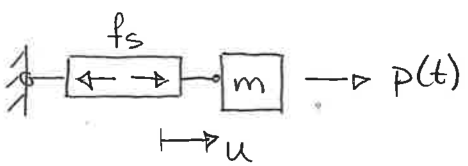
STRUCTURAL DYNAMIC COMPUTING

SDOF systems damping included:

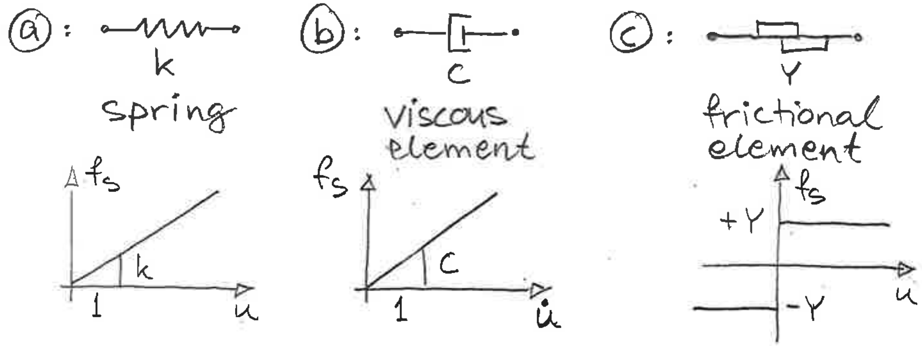
- Various damping and hysteresis in harmonic loading
- Equation of motion – standard form
- Free vibrations - damped systems
- Forced harmonic vibration with damping
- Complex representation - damped systems
- Dynamic response factor
- Introduction to time stepping methods

SDOF SYSTEMS WITH DAMPING INCLUDED

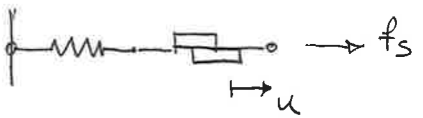


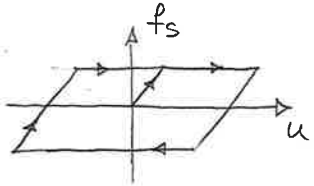
$$\boxed{\text{General}: \quad f_s = f_s(u, \dot{u})}$$

The force f_s can be represented by simple elements like:

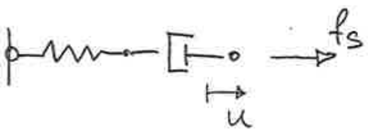


<u>Ex</u> Ideal plasticity

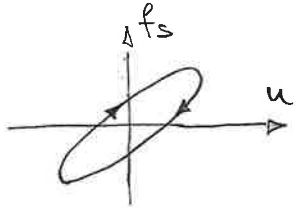




Viscous fluid



(Maxwell model)



Response to cyclic load

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SDOF SYSTEMS WITH ... cont.

The elements ⓐ, ⓑ and ⓒ provides easy understandable mechanical analogies ...

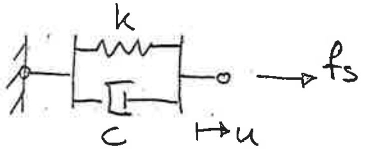
However, only combinations of ⓐ and ⓑ yields <u>linear dynamic systems</u>!

Arbitrary combinations of ⓐ and ⓑ has the property:

$$p(t) = p_0 \sin \omega t \rightsquigarrow u(t) = u_0 \sin(\omega t - \phi)$$

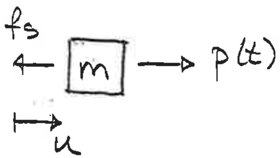
$$\boxed{\text{Harmonic loading} \Rightarrow \text{Harmonic response with the same angular frequency } \omega}$$

The <u>Kelvin model</u> is a particularly simple viscous solid model:



$$\boxed{f_s = ku + c\dot{u}} \quad \ldots (1)$$

Adding the mass:

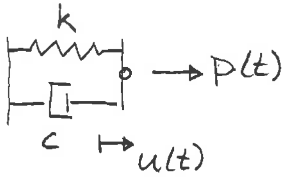


$$p(t) - f_s = m\ddot{u} \quad \ldots (2)$$

(1) and (2) $\Rightarrow$ $\boxed{m\ddot{u} + c\dot{u} + ku = p(t)}$

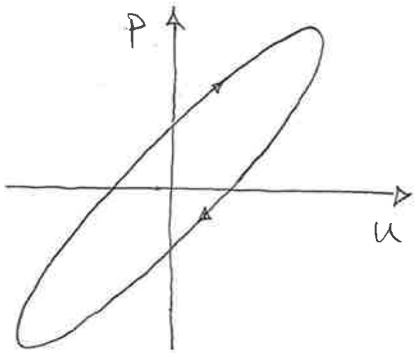
HYSTERESIS IN HARMONIC LOADING

Linear dynamic structural response without inertia (i.e. mass excluded):



$p(t) = p_0 \sin \omega t \quad \leadsto \quad u(t) = u_0 \sin(\omega t - \phi)$

phase angle



> The force-displacement curve is elliptical and the area inside represent losses in the structure.

Arbitrary combinations of linear springs and dashpots yields linear dynamic response.

Hysteresis cont.

Energy dissipated by viscous damping in one cycle* :

$$E_D = 2\pi \xi \frac{\omega}{\omega_n} k u_0^2$$

External force p(t) energy input* :

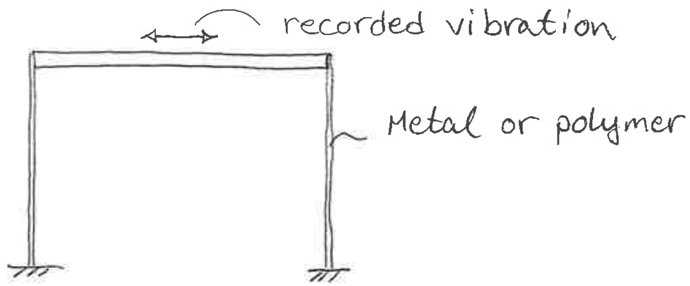
$$E_I = \pi p_0 u_0 \sin \phi$$

In steady state :

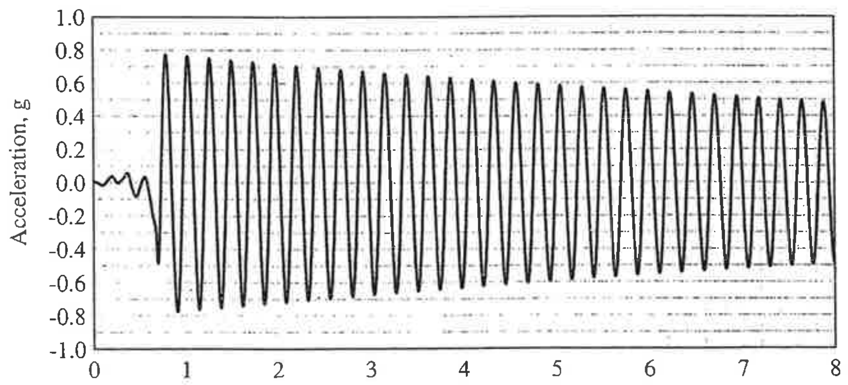
$$E_D = E_I$$

Rem. *) Derivations in section 3.8, p. 99-101

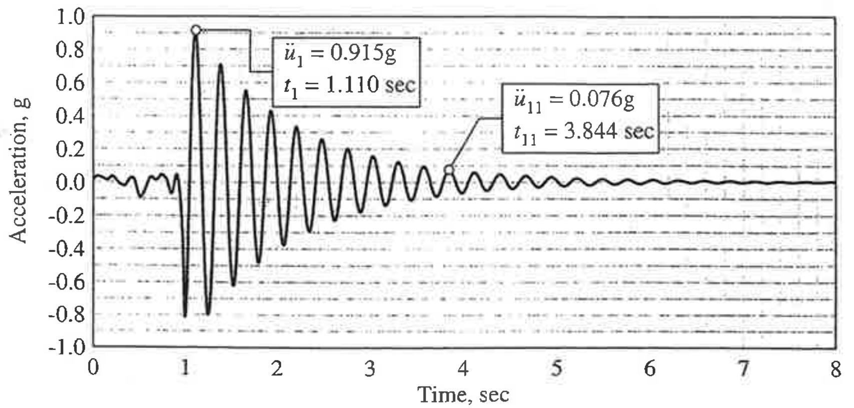
DAMPING OF FREE VIBRATIONS



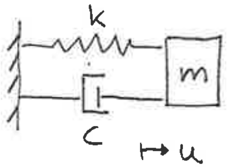
Aluminum :



Plexiglass :



VISCOUSLY DAMPED FREE VIBRATIONS



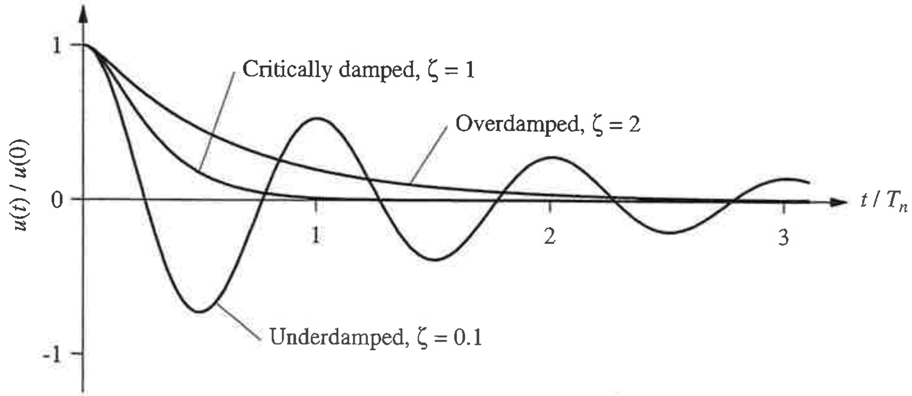
SDOF system

Eq. of motion: $m\ddot{u} + c\dot{u} + ku = 0$

Rewrite the equation ↝

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0$$

with $\omega_n = \sqrt{\frac{k}{m}}$ and $\xi = \frac{c}{2m\omega_n}$



$$u(t) = e^{-\xi\omega_n t}\left[u(0)\cos\omega_D t + \frac{\dot{u}(0) + \xi\omega_n u(0)}{\omega_D}\sin\omega_D t\right]$$

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

valid for $\xi < 1$

Exponentially decaying harmonic function.

Ex. 

NII (→) $\quad -c\dot{u} - ku = m\ddot{u}$

$$m\ddot{u} + c\dot{u} + ku = 0$$

Rewrite in standard form:

$$\ddot{u} + \underbrace{\frac{c}{m}}_{2\xi\omega_n}\dot{u} + \underbrace{\frac{k}{m}}_{\omega_n^2}u = 0 \longrightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}}$$

$$2\xi\omega_n = \frac{c}{m} \; ; \; \underline{\xi = \frac{c}{2m\omega_n}}$$

Free vibration solution with initial value $u(0) = u_0 \quad \dot{u}(0) = 0$:

$$u(t) \approx e^{-\xi\omega_n t} u_0 (\cos\omega_n t + \xi \sin\omega_n t)$$

if $\xi \lessapprox 0.2 \qquad \underline{\omega_n = \omega_D}$

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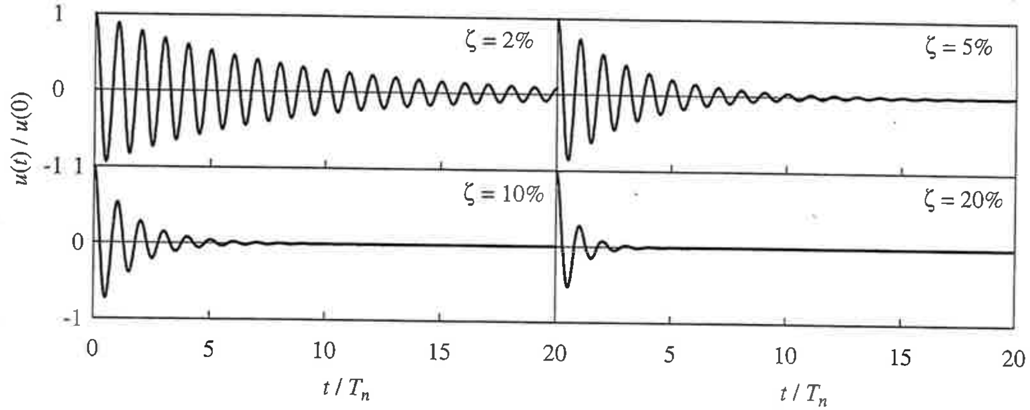
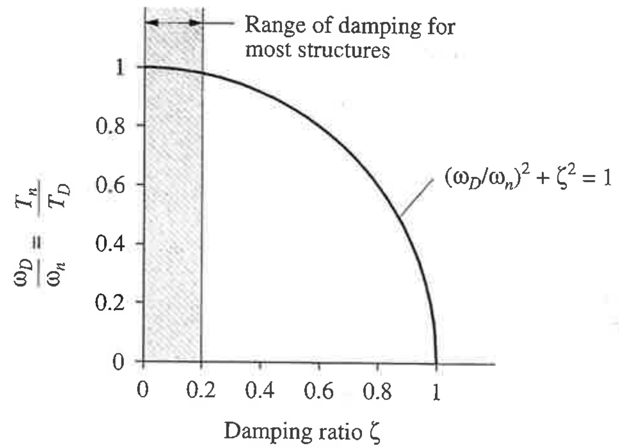
Rem. ξ is a dimensionless damping parameter.

Various behaviours for realistic levels of damping:



The natural frequency is not influenced very much by moderate viscous damping (i.e. $\xi < 0.2$):



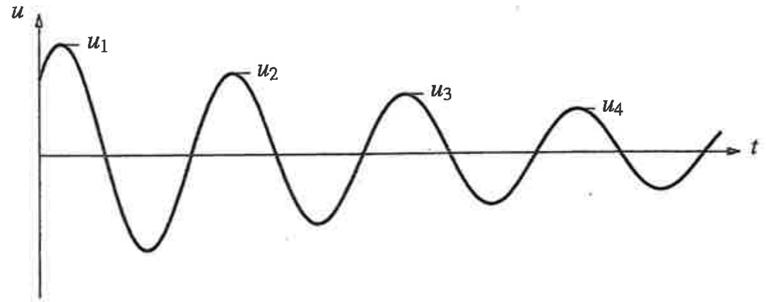
$$\Rightarrow \quad \underline{\omega_D \approx \omega_n}$$

EVALUATION OF DAMPING - FREE VIB.

For lightly damped systems the damping ratio ξ can be determined from :

$$\xi = \frac{1}{2\pi j} \ln \frac{u_i}{u_{i+j}} \quad \text{or} \quad \xi = \frac{1}{2\pi j} \ln \frac{\ddot{u}_i}{\ddot{u}_{i+j}}$$

ie by using displacements ...



or accelerations decay ...

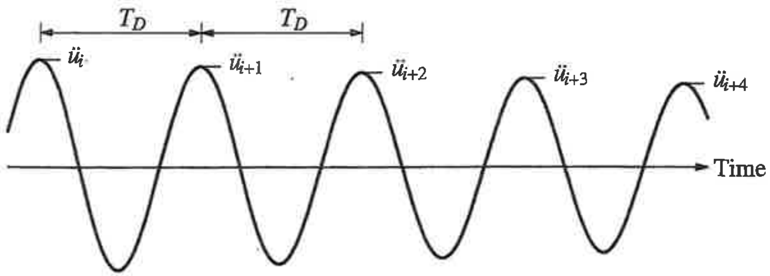


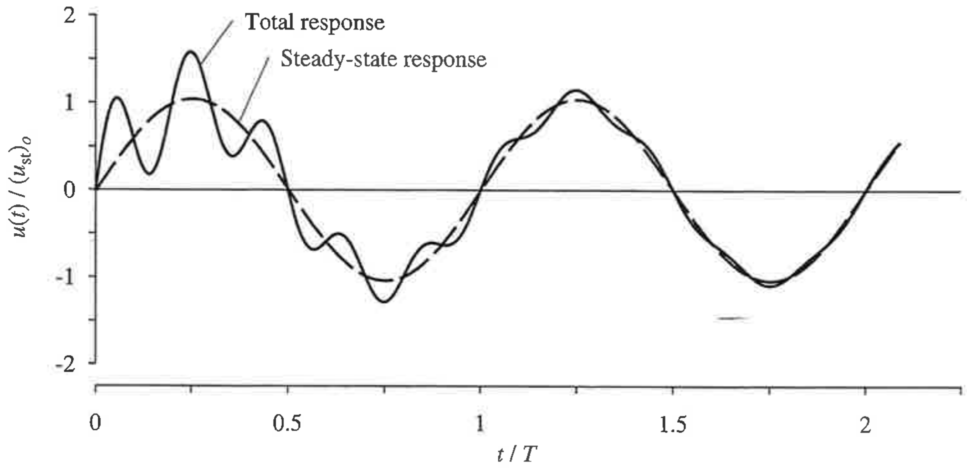
Figure 2.2.8 Acceleration record of a freely vibrating system.

Example 2.5 p. 55 uses 10 cycles ($j = 10$) for evaluation of ξ.

FORCED VIBRATIONS WITH VISCOUS DAMPING

Eq. of motion : $m\ddot{u} + c\dot{u} + ku = p_0 \sin \omega t$

The homogeneous solution fades away and the particular solution = steady state solution remains.



The steady state displacement solution is phase shifted compared to the driving force :

$$\begin{cases} \frac{\omega}{\omega_n} < 1 & \text{in phase} \\ \frac{\omega}{\omega_n} > 1 & \text{out of phase} \end{cases}$$

LINEAR DYNAMIC RESPONSE TO HARMONIC LOADING OF A DAMPED STRUCTURE

Linear dynamic response to a force

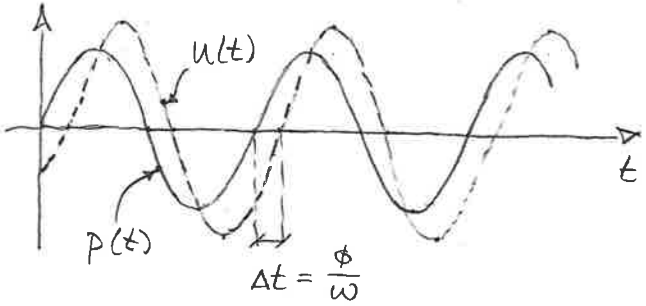
$$p(t) = p_o \sin \omega t$$

is a delayed displacement

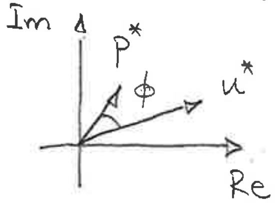
$$u(t) = u_o \sin(\omega t - \phi)$$

with the same frequency.

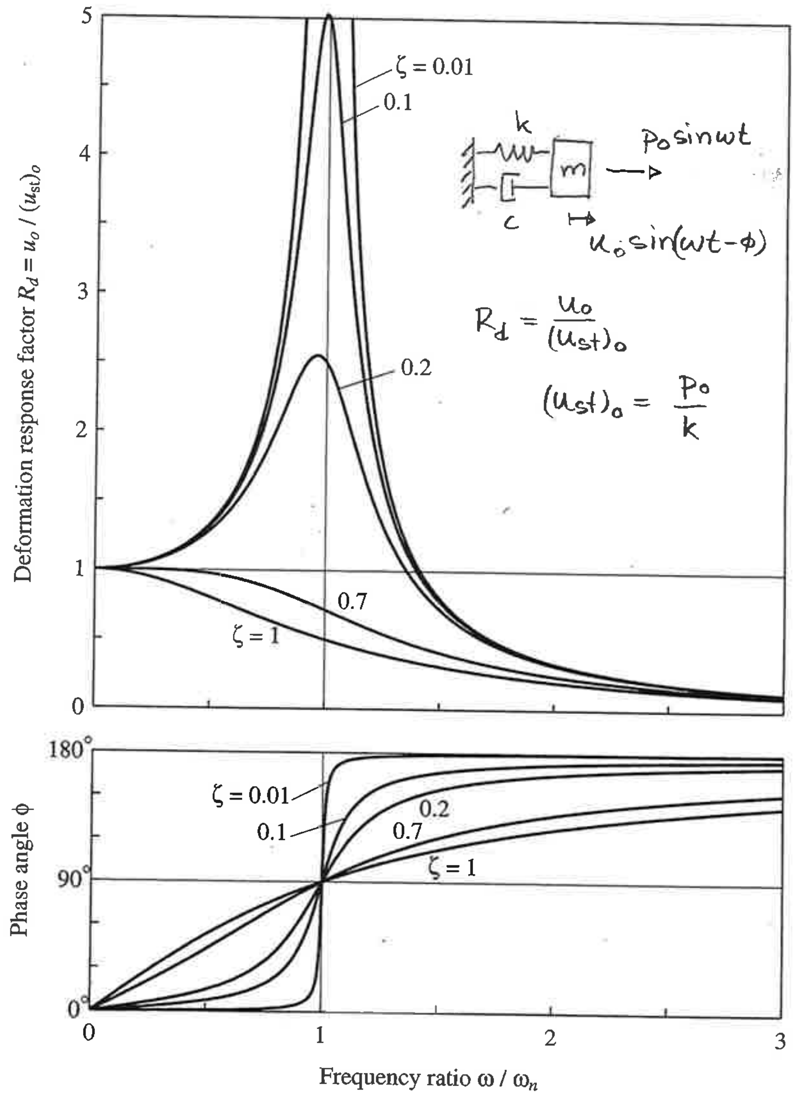
The phase angle ϕ determines the delay in the response.



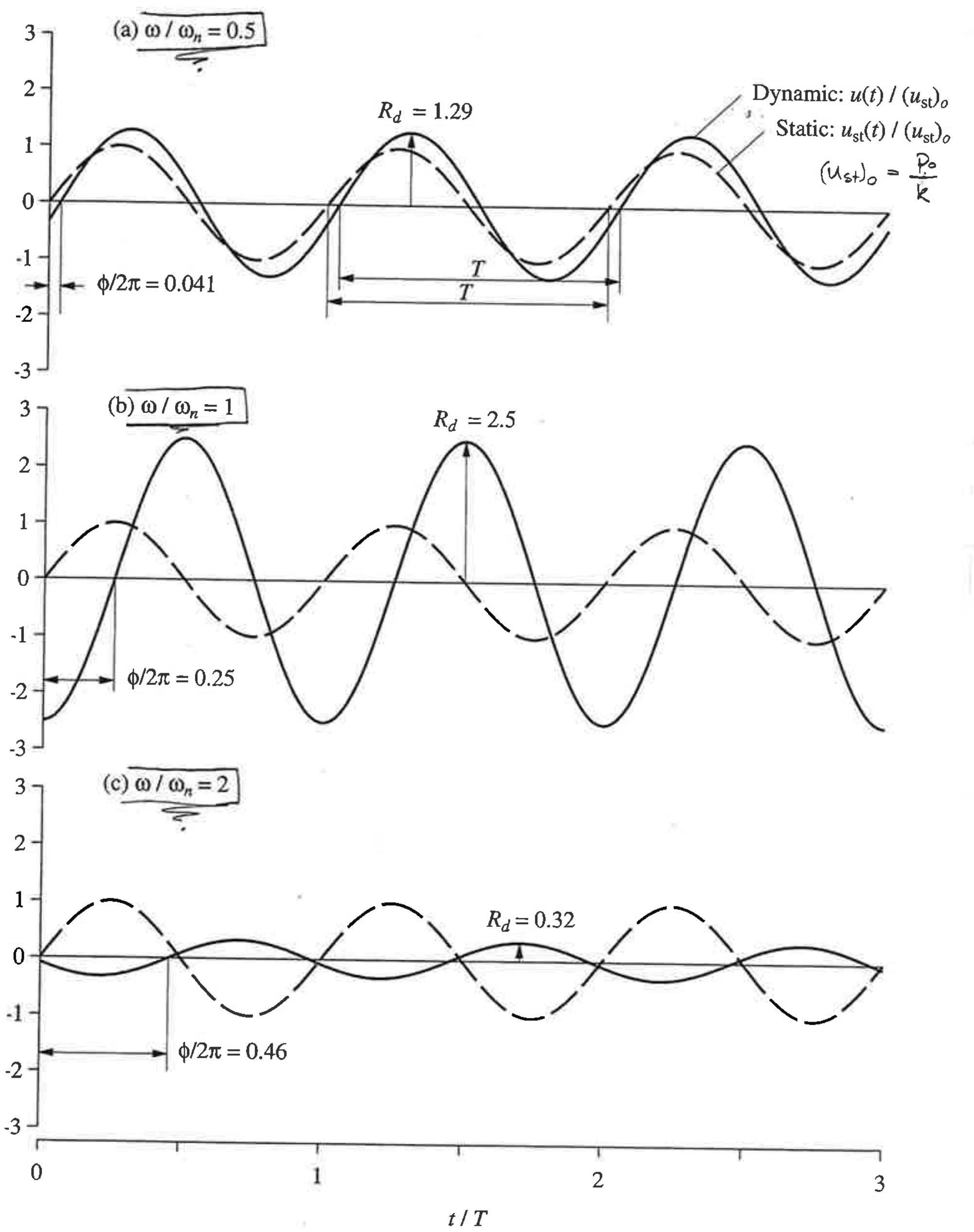
In a rotating vector diagram:



Response in terms of amplitude u_o and phase angle ϕ:



Time function normalized responses: ($\xi = 0.2$)
---- force ——— displacement



Displacement amplitude:

1. If the frequency ratio $\omega/\omega_n \ll 1$ (i.e., $T \gg T_n$, implying that the force is "slowly varying"), R_d is only slightly larger than 1 and is essentially independent of damping. Thus

$$u_o \simeq (u_{st})_o = \frac{p_o}{k} \tag{3.2.13}$$

This result implies that the amplitude of dynamic response is essentially the same as the static deformation and is controlled by the stiffness of the system.

2. If $\omega/\omega_n \gg 1$ (i.e., $T \ll Tn$, implying that the force is "rapidly varying"), R_d tends to zero as ω/ω_n increases and is essentially unaffected by damping. For large values of ω/ω_n, the $(\omega/\omega_n)^4$ term is dominant in Eq. (3.2.11), which can be approximated by

$$u_o \simeq (u_{st})_o \frac{\omega_n^2}{\omega^2} = \frac{p_o}{m\omega^2} \tag{3.2.14}$$

This result implies that the response is controlled by the mass of the system.

3. If $\omega/\omega_n \simeq 1$ (i.e., the forcing frequency is close to the natural frequency of the system), R_d is very sensitive to damping and, for the smaller damping values, R_d can be several times larger than 1, implying that the amplitude of dynamic response can be much larger than the static deformation. If $\omega = \omega_n$, Eq. (3.2.11) gives

$$u_o = \frac{(u_{st})_o}{2\zeta} = \frac{p_o}{c\omega_n} \tag{3.2.15}$$

This result implies that the response is controlled by the damping of the system.

Phase angle:

The phase angle ϕ, which defines the time by which the response lags behind the force, varies with ω/ω_n as shown in Fig. 3.2.6. It is examined next for the same three regions of the excitation-frequency scale:

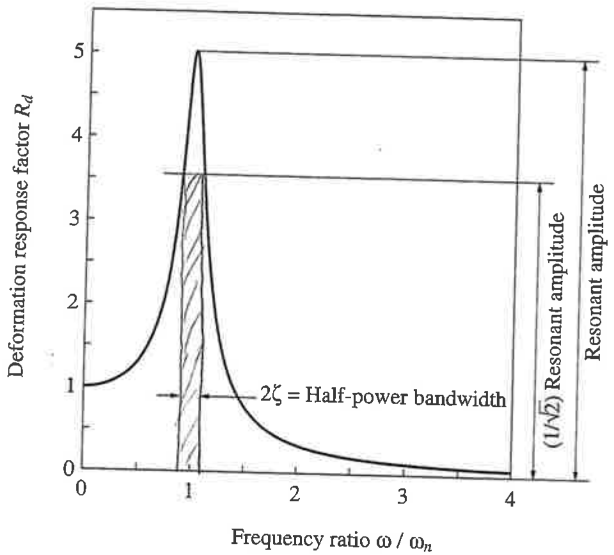
1. If $\omega/\omega_n \ll 1$ (i.e., the force is "slowly varying"), ϕ is close to 0° and the displacement is essentially in phase with the applied force, as in Fig. 3.2.5a. When the force in Fig. 1.2.1a acts to the right, the system would also be displaced to the right.

2. If $\omega/\omega_n \gg 1$ (i.e., the force is "rapidly varying"), ϕ is close to 180° and the displacement is essentially out of phase relative to the applied force, as in Fig. 3.2.5c. When the force acts to the right, the system would be displaced to the left.

3. If $\omega/\omega_n = 1$ (i.e., the forcing frequency is equal to the natural frequency), $\phi = 90°$ for all values of ζ, and the displacement attains its peaks when the force passes through zeros, as in Fig. 3.2.5b.

HALF POWER BANDWITH - EVALUATION OF DAMPING PROPERTIES

The deformation response factor ...



... provides information of the damping of the system.
Evaluating at $\frac{1}{\sqrt{2}}$ of the resonant amplitude (half power) for small damping ξ yields*:

$$\frac{\omega_b - \omega_a}{\omega_n} \approx 2\xi \quad ; \quad \xi = \frac{\omega_b - \omega_a}{2\omega_n}$$

Damping can be determined for a structure from a forced vibration test!

<u>Rem.</u> *) see derivation 3.4 p. 84

COMPLEX REPRESENTATION

Eulers formula :

$$u^* = u_0 e^{i\omega t} = u_0 \cos\omega t + i u_0 \sin\omega t$$

Time derivative :

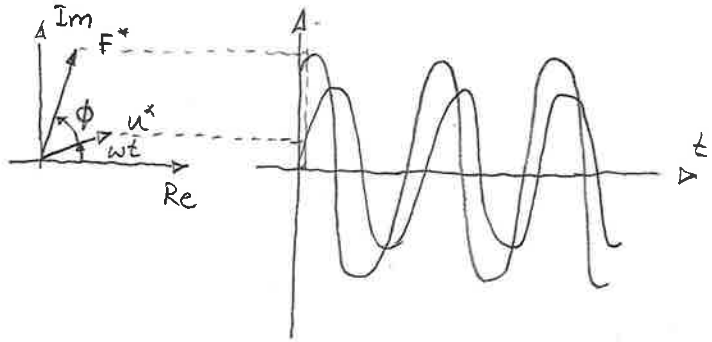
$$\dot{u}^* = \frac{du^*}{dt} = i\omega u_0 e^{i\omega t} = i\omega u^* \Rightarrow$$

$$\boxed{\frac{du^*}{dt} = i\omega u^*}$$

Differential equations $\rightsquigarrow$ algebraic eq.

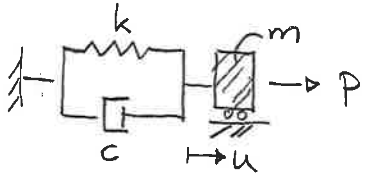
> Use of complex representation is convenient for calculations

In damped structurs force and displacement are phase shifted (ϕ) :



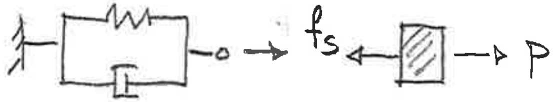
Steady state response is conveniently analysed with complex calculus.

FORCED DAMPED VIBRATION - COMPLEX ANALYSIS



Steady state sinusoidal

$p = p_0 \sin\omega t$



Spring and damper in parallell :

$$f_s = ku + c\dot{u} \quad \ldots (1)$$

NII for the mass ($\rightarrow$): $m\ddot{u} = p - f_s \quad \ldots (2)$

Complex: $u \rightsquigarrow u^*$, $\dot{u} \rightsquigarrow i\omega u^*$,

$p \rightsquigarrow p^*$, $f_s \rightsquigarrow f_s^*$

Complex stiffness: $f_s^* = \underbrace{(k + i\omega c)}_{k^*} u^* \quad \ldots (1')$

NII $\Rightarrow$ $(i\omega)^2 m u^* = p^* - f_s^* \quad \ldots (2')$

(1') into (2') $\Rightarrow$

$$-\omega^2 m u^* = p^* - k^* u^* \; ; \quad \boxed{u^* = \frac{p^*}{k^* - \omega^2 m}}$$

$$\begin{cases} \text{Amplitude} & u_0 = |u^*| \\ \text{Phase} & \phi = \arg(u^*) \end{cases}$$

__Amplitude__: Put $p^* = p_0$ as a real variable

$$u_0 = |u^*| \; ; \quad u_0 = \frac{p_0}{|k^* - \omega^2 m|} \; ;$$

$$u_0 = \frac{p_0}{|k - \omega^2 m + i\omega|} \; ; \quad u_0 = \frac{p_0}{\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}}$$

① $\omega \ll \omega_n$, ω^2 small $\Rightarrow$

$u_0 \approx \dfrac{p_0}{k}$ "static" stiffness controlled amplitude

② $\omega \gg \omega_n$, ω^4 dominating $\Rightarrow$

$u_0 \approx \dfrac{p_0}{\omega^2 m}$ mass controlled small amplitude

③ $\omega = \omega_n$, $u_0 = \dfrac{p_0}{\omega_n c}$

damping controlled large amplitude

__Phase__: ① In phase $\phi \approx 0°$

② Out of phase $\phi \approx 180°$

③ $\phi = 90°$

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Alternatively by using the normalized equation of motion (using $\xi = \frac{c}{2m\omega_n}$):

$$\ddot{u} + 2\xi\omega_n \dot{u} + \omega_n^2 u = \frac{p_0}{m} \sin \omega t$$

Complex form:

$$(i\omega)^2 u^* + 2\xi\omega_n i\omega u^* + \omega_n^2 u^* = \frac{p^*}{m} \; ;$$

$$u^* (\omega_n^2 - \omega^2 + i\, 2\xi\omega_n\omega) = \frac{p^*}{m} \; ;$$

$$u^* \left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\, 2\xi \frac{\omega}{\omega_n}\right) = \frac{p^*}{\omega_n^2 m} \; ;$$

Assuming $p^* = p_0$ real and using $\omega_n^2 = \frac{k}{m}$ $\Rightarrow$

$$u^* = \frac{p_0}{k} \underbrace{\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\, 2\xi \left(\frac{\omega}{\omega_n}\right)}}$$

Complex dynamic response faktor R_d^*, dimensionless in $\left(\frac{\omega}{\omega_n}\right)$.

$$\begin{cases} R_d = \left|\dfrac{u^*}{p_0/k}\right| & \text{the magnitude of } R_d^* \\ -\phi = -\arg\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\, 2\xi \left(\frac{\omega}{\omega_n}\right)\right) & \text{phase shift} \end{cases}$$

$\left(\dfrac{p_0}{k} = (u_{st})_0 \quad \text{static solution for } p_0 \right)$

__Rem*__: Def. phase $\sin(\omega t - \phi)$

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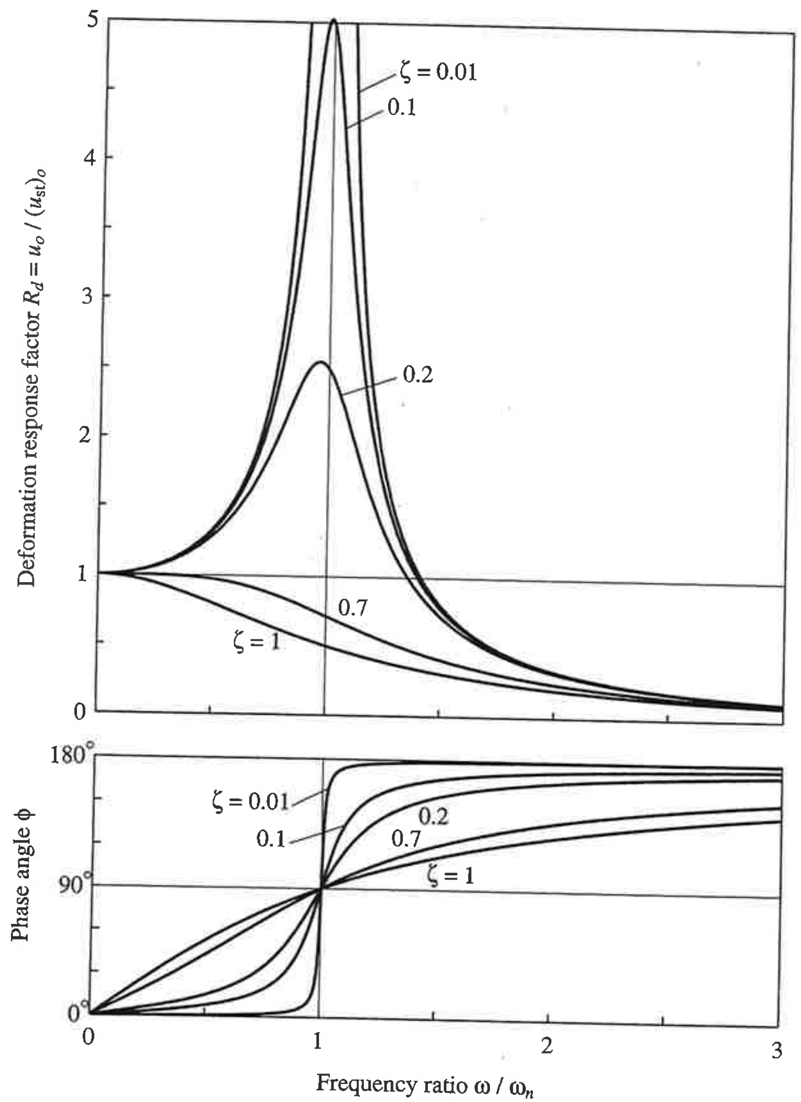
FORCED DAMPED VIBRATIONS cont.

Deformation response factor R_d



TIME-STEPPING METHODS – INTRODUCTION

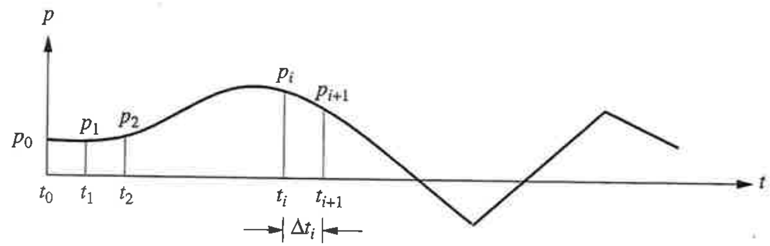
Consider

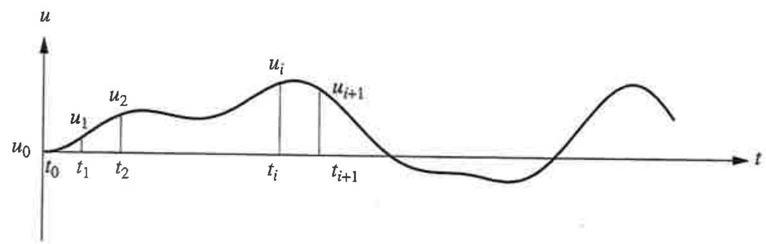
$$\begin{cases} m\ddot{u} + c\dot{u} + f_s(u, \dot{u}) = p(t) \\ u_0 = u(0) \quad \text{and} \quad \dot{u}_0 = \dot{u}(0) \end{cases}$$

Determine response quantities

$$u_i, \ \dot{u}_i \quad \text{and} \quad \ddot{u}_i$$

at discrete time instants t_i





These values satisfy

$$m\ddot{u}_i + c\dot{u}_i + (f_s)_i = p_i$$

Procedures giving u_{i+1}, $\dot{u}_{i+1}$ and $\ddot{u}_{i+1}$ at t_{i+1} will be discussed.

A SIMPLE TIME STEPPING PROCEDURE

Solve $m\ddot{u} + c\dot{u} + ku = p(t)$

with $\begin{cases} u(0) = u_0 \\ \dot{u}(0) = v_0 \end{cases}$

Start by determining the acceleration at $t = 0$:

$$a_0 \equiv \ddot{u}(0) = \frac{1}{m}\left(p(0) - c v_0 - k u_0\right)$$

By assuming constant acceleration in time steps Δt a simple procedure is obtained :

$$v_1 \equiv \dot{u}(\Delta t) \approx a_0 \Delta t + v_0$$

$$u_1 \equiv u(\Delta t) \approx a_0 \frac{(\Delta t)^2}{2} + v_0 \Delta t + u_0$$

An algorithm can now be established :

for $i = 1$ to n

$a_i = \frac{1}{m}\left(p(t_i) - c v_i - k u_i\right)$

$v_{i+1} = a_i \Delta t + v_i$

$u_{i+1} = a_i (\Delta t)^2/2 + v_i \Delta t + u_i$

next i

This is a special (not very accurate) case of the Newmark family of time stepping procedures, implemented in CALFEM.