Transient Heat Flow

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Contents

1. One dimensional transient heat flow

1.1. Introduction

This text is intended to provide an extension to "Introduction to the Finite Element Method" by Ottosen and Petersson by introducing to the subject of transient flow problems. The presentation follows that book as close as possible in notation and in the derivations of the finite element equations.

1.2. One dimensional transient heat equation - strong form

At stationary heat conduction the amount of heat supplied to the body per unit time equals the amount of heat leaving the body per unit time. It is the implied that there is no variation of temperature or heat flow with time in a *fixed point* in space, although there is a variation from point to point in space. The balance equation for a one-dimensional heat flow problem at stationary conditions may be written as

$$-\frac{d}{dx}(Aq) + Q = 0 \tag{1.1}$$

where A [m^2] is the cross-sectional area, q [J/sm^2] the heat flux and Q [J/sm] heat supply.

Transient heat conduction, however, implies that the temperature and heat flow, in a fixed point is a function of time, i.e. $q = q(x,t)$ and $T = T(x,t)$ also the heat source and the material properties may be functions of time as $Q = Q(x,t)$ and $k = k(x,t)$. Since the amount of heat supplied to the body per unit time *not* equals the amount of heat leaving the body per unit time the temperature will change in the body. The rate of temperature change, however, depends on the *heat capacity* $c(x,t)$, which is a material property that states the resistance to temperature change. The heat capacity is a scalar parameter that quantifies the amount of energy required, per unit mass, to rise the temperature one degree, i.e. in SI units $J/(K \cdot kg)$. The balance equation (1.1) for the steady state problem may for the case of transient heat conduction be written as

$$\boxed{-\frac{d}{dx}(Aq) + Q = \rho A c \frac{dT}{dt}} \tag{1.2}$$

where ρ is the density of the material. By inserting the constitutive relation for one dimensional heat conduction, *Fourier's law*, that is written

$$\boxed{q = -k\frac{dT}{dx}} \tag{1.3}$$

the following differential equation is formulated

$$\frac{d}{dx}(Ak\frac{dT}{dx}) + Q = \rho Ac\frac{dT}{dt}; \quad 0 \leq x \leq L \tag{1.4}$$

k is the thermal conductivity. The *one-dimensional transient heat equation* established applies for the region considered. To solve the differential equation a region and boundary conditions must be specified. Two boundary conditions must be specified; one for each end of the one dimensional body. We may assume one end to have a prescribed temperature and at the other end a prescribed flux. The strong form of one-dimensional transient heat flow may now be formulated.

Strong form of one-dimensional transient heat flow

$$\begin{aligned}
&\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = \rho Ac\frac{dT}{dt}; \quad 0 \leq x \leq L \\
&q(x=0) = -\left(k\frac{dT}{dx}\right)_{x=0} = h \\
&T(x=L) = g
\end{aligned} \tag{1.5}$$

1.3. Weak form of one-dimensional transient heat flow

The strong form of one-dimensional transient heat flow may be reformulated into a weak form in the same manner as for the steady state heat flow. The weak form is established by multiply (1.5) with an arbitrary time-independent weight function $v(x)$ and integrating over the region.

$$\int_0^L v\left[\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q\right] dx = \int_0^L v\rho Ac\frac{dT}{dt}\, dx \tag{1.6}$$

The first term in this equation may be integrated by parts as

$$\int_0^L v\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)\, dx = \left[vAk\frac{dT}{dx}\right]_0^L - \int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}\, dx \tag{1.7}$$

Use of this expression in (1.6) implies that

$$\int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}\, dx + \int_0^L v\rho Ac\frac{dT}{dt}\, dx = \left[vAk\frac{dT}{dx}\right]_0^L + \int_0^L vQ\, dx \tag{1.8}$$

Now, using that $q = -kdT/dx$ and inserting the natural boundary condition $q(x = 0) = h$, the boundary term may be rewritten to the final weak form of one dimensional transient heat flow.

Weak form of one-dimensional transient heat flow

$$\int_0^L \frac{dv}{dx} Ak \frac{dT}{dx}\,dx + \int_0^L v\rho Ac \frac{dT}{dt}\,dx = -(vAq)_{x=L} + (vA)_{x=0}h + \int_0^L vQ\,dx$$

$$T(x = L) = g \tag{1.9}$$

The main difference of the weak form of transient heat flow compared to the corresponding steady state case is that we now have derivatives both with respect to the spatial coordinates and to the time. This implies that approximations for the temperature must be made in both spatial coordinates and the time.

1.4. Spatial approximation of one-dimensional transient heat flow

As for the case of steady state heat flow an approximation of the temperature is introduced. The approximation may for the transient case be separated as

$$T(x, t) = \mathbf{N}(x)\mathbf{a}(t) \tag{1.10}$$

where $\mathbf{N}(x)$ is the shape functions that are functions of x and $\mathbf{a}(t)$ is the nodal temperatures that are functions of time. The weak form (1.9) contain both dT/dx and dT/dt. Since $\mathbf{a}$ is independent of x we may write

$$\frac{dT}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{a} = \mathbf{B}\mathbf{a} \tag{1.11}$$

and since $\mathbf{N}$ is independent of t we may write

$$\frac{dT}{dt} = \mathbf{N}\frac{d\mathbf{a}}{dt} = \mathbf{N}\dot{\mathbf{a}} \tag{1.12}$$

Adopting the Galerkin method the scalar weight function v is chosen as

$$v = \mathbf{N}\mathbf{c} = \mathbf{c}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}} \tag{1.13}$$

and

$$\frac{dv}{dx} = \mathbf{c}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}} \quad \text{where} \quad \mathbf{B}^{\mathbf{T}} = \frac{d\mathbf{N}^{\mathbf{T}}}{dx} \tag{1.14}$$

The approximation (1.10) and the choice of the weight function (1.13) is inserted in the weak form of the one dimensional transient heat flow (1.8) and also using the fact that the vector $\mathbf{c}$ is independent of x results in

$$\mathbf{c}^{\mathbf{T}}\left(\int_0^L \mathbf{B}^{\mathbf{T}} Ak\,\mathbf{B}\,dx\;\mathbf{a} + \int_0^L \mathbf{N}^{\mathbf{T}}\rho Ac\mathbf{N}\,dx\;\dot{\mathbf{a}} + \left[\mathbf{N}^{\mathbf{T}}Aq\right]_0^L - \int_0^L \mathbf{N}^{\mathbf{T}}Q\;dx\right) = 0 \tag{1.15}$$

where the vector $\mathbf{a}$ now contain the nodal temperatures as function of the time and $\dot{\mathbf{a}}$ contain time derivatives of the nodal temperatures. As this expression is valid for arbitrary $\mathbf{c}^{\mathbf{T}}$ vectors it is concluded that

$$\int_0^L \mathbf{B}^{\mathbf{T}} Ak\,\mathbf{B}\,dx\;\mathbf{a} + \int_0^L \mathbf{N}^{\mathbf{T}}\rho Ac\mathbf{N}\,dx\;\dot{\mathbf{a}} = -\left[\mathbf{N}^{\mathbf{T}}Aq\right]_0^L + \int_0^L \mathbf{N}^{\mathbf{T}}Q\;dx \tag{1.16}$$

which also may be written as

$$\boxed{\mathbf{K}\mathbf{a} + \mathbf{C}\dot{\mathbf{a}} = \mathbf{f_b} + \mathbf{f_l}} \tag{1.17}$$

where

$$\boxed{\begin{aligned}
\mathbf{K} &= \int_0^L \mathbf{B}^{\mathbf{T}} Ak\,\mathbf{B}\,dx \\
\mathbf{C} &= \int_0^L \mathbf{N}^{\mathbf{T}}\rho Ac\mathbf{N}\,dx \\
\mathbf{f_b} &= -\left[\mathbf{N}^{\mathbf{T}}Aq\right]_0^L \\
\mathbf{f_l} &= \int_0^L \mathbf{N}^{\mathbf{T}}Q\;dx
\end{aligned}} \tag{1.18}$$

Defining the force vector $\mathbf{f}$ as

$$\mathbf{f} = \mathbf{f_b} + \mathbf{f_l} \tag{1.19}$$

equation 1.17 may be written in a compact form as

$$\boxed{\mathbf{K}\mathbf{a} + \mathbf{C}\dot{\mathbf{a}} = \mathbf{f}} \tag{1.20}$$

Since the vector $\mathbf{a}$ contain the nodal temperatures as function of the time and $\dot{\mathbf{a}}$ contain time derivatives of the nodal temperatures (1.17) represents a system of ODE's (ordinary differential equations) of order 1 and an approximation in time must also be made. The matrix $\mathbf{C}$ is called the *capacity* matrix and is the only term that differs from the steady state case.

EXAMPLE 1 - Calculation of element capacity matrix $\mathbf{C}^e$

A linear one-dimensional spring element of length L is assumed. Moreover, the area, A, the density ρ and the capacitivity c is assumed to be constant. The shape functions for a linear spring element are given by

$$N_1^e = 1 - \frac{x}{L} \quad N_2^e = \frac{x}{L} \tag{1.21}$$

The capacitivity matrix is according to equation (1.16) given by

$$\mathbf{C}^e = \int_0^L \mathbf{N}^{eT} \rho A c \mathbf{N}^e dx = \rho A c \begin{bmatrix} \int_0^L N_1^e N_1^e dx & \int_0^L N_1^e N_2^e dx \\ \int_0^L N_2^e N_1^e dx & \int_0^L N_2^e N_2^e dx \end{bmatrix} \tag{1.22}$$

Inserting the element shape functions we arrive at

$$\mathbf{C}^e = \rho A c \begin{bmatrix} \int_0^L 1 - 2\frac{x}{L} + \frac{x^2}{L^2} dx & \int_0^L \frac{x}{L} - \frac{x^2}{L^2} dx \\ \int_0^L \frac{x}{L} - \frac{x^2}{L^2} dx & \int_0^L \frac{x^2}{L^2} dx \end{bmatrix} \tag{1.23}$$

Integrating results in

$$\mathbf{C}^e = \rho A L c \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \tag{1.24}$$

1.5. Approximation in time of one-dimensional transient heat flow

The finite element equations now have to be approximated in time. Different choices of the time approximation may be made but here only a linear time approximation will be considered. The time is divided into a certain number of time-steps Δt. The discretization is assumed to start at the time $\tau = 0$ where τ is the time coordinate, see Figure 1.1.

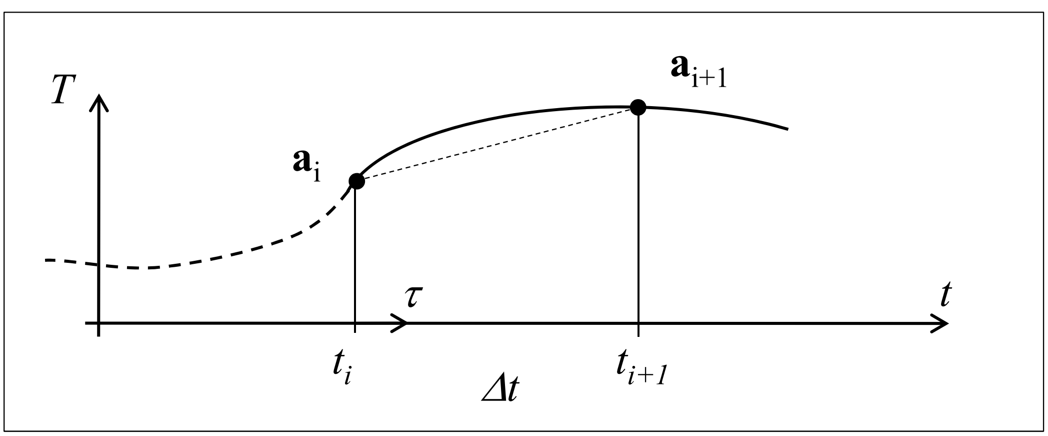


Figure 1.1: *Integration in time.*

Since the temperatures is assumed to vary linearly within each time-step, the variation of the temperature may according to Figure 1.1 be written as

$$\mathbf{a}(\tau) = \frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t}\tau + \mathbf{a}_i \tag{1.25}$$

This approximation in time may also be written in terms of time functions (c.f. shape functions) as

$$\mathbf{a}(\tau) = G_i\mathbf{a}_i + G_{i+1}\mathbf{a}_{i+1} \tag{1.26}$$

where G_i and G_{i+1} are the linear time functions that according to (1.25) are

$$G_i = 1 - \frac{\tau}{\Delta t}; \quad G_{i+1} = \frac{\tau}{\Delta t} \tag{1.27}$$

The derivative of the temperature in time may then immediately be written

$$\dot{\mathbf{a}}(\tau) = -\frac{1}{\Delta t}\mathbf{a}_i + \frac{1}{\Delta t}\mathbf{a}_{i+1} = \frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t} \tag{1.28}$$

The temperatures $\mathbf{a}_i$ at the time t_i is now assumed to be known and it is then the temperatures $\mathbf{a}_{i+1}$ at the time t_{i+1} that is to be determined. To introduce this choice of the time approximation, the FE-formulation given by (1.17) is multiplied with a weight function in time $w(\tau)$ and integrated over the time-step

$$\int_0^{\Delta t} w(\mathbf{Ka} + \mathbf{C}\dot{\mathbf{a}})\, d\tau = \int_0^{\Delta t} w\mathbf{f}\, d\tau \tag{1.29}$$

the integrals may be separated

$$\int_0^{\Delta t} w\mathbf{C}\dot{\mathbf{a}}\, d\tau + \int_0^{\Delta t} w\mathbf{Ka}\, d\tau = \int_0^{\Delta t} w\mathbf{f}\, d\tau \tag{1.30}$$

the time approximation given in (1.25) and (1.28) is now inserted in (1.31)

$$\int_0^{\Delta t} w\mathbf{C}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t}\, d\tau + \int_0^{\Delta t} w\mathbf{K}\left[\mathbf{a}_i + \frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t}\tau\right] d\tau = \int_0^{\Delta t} w\mathbf{f}\, d\tau \tag{1.31}$$

Since $\mathbf{C}$, $\mathbf{K}$, $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$ are independent of time

$$\mathbf{C}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t}\int_0^{\Delta t} w\, d\tau + \mathbf{K}\mathbf{a}_i\int_0^{\Delta t} w\, d\tau + \mathbf{K}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t}\int_0^{\Delta t} w\tau\, d\tau = \int_0^{\Delta t} w\mathbf{f}\, d\tau \tag{1.32}$$

This equation may now be divided by $\int_0^{\Delta t} w d\tau$ which results in

$$\mathbf{C}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t} + \mathbf{K}\mathbf{a}_i + \mathbf{K}\frac{\int_0^{\Delta t} w\tau\, d\tau}{\int_0^{\Delta t} w\, d\tau}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t} = \frac{\int_0^{\Delta t} w\mathbf{f}\, d\tau}{\int_0^{\Delta t} w\, d\tau} \tag{1.33}$$

Introducing Θ as the weighting parameter as

$$\Theta = \frac{1}{\Delta t} \frac{\displaystyle\int_0^{\Delta t} w\tau \, d\tau}{\displaystyle\int_0^{\Delta t} w \, d\tau} \tag{1.34}$$

equation (1.33) may be written as

$$\boxed{\mathbf{C}\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{\Delta t} + \mathbf{K}\left[\mathbf{a}_i + \Theta\left(\mathbf{a}_{i+1} - \mathbf{a}_i\right)\right] = \bar{\mathbf{f}}} \tag{1.35}$$

where $\bar{\mathbf{f}}$ represents an average value of $\mathbf{f}$ and is given by

$$\bar{\mathbf{f}} = \frac{\displaystyle\int_0^{\Delta t} w\mathbf{f} \, d\tau}{\displaystyle\int_0^{\Delta t} w \, d\tau} \tag{1.36}$$

this load vector may also be assumed to vary linear in time and may then according to (1.26) be written as

$$\mathbf{f}(\tau) = G_i \mathbf{f}_i + G_{i+1}\mathbf{f}_{i+1} = (1 - \frac{\tau}{\Delta t})\mathbf{f}_i + \frac{\tau}{\Delta t}\mathbf{f}_{i+1} \tag{1.37}$$

The final form of the force vector may now be determined by inserting equation (1.37) in (1.36) to get

$$\bar{\mathbf{f}} = \mathbf{f}_i + \Theta(\mathbf{f}_{i+1} - \mathbf{f}_i) \tag{1.38}$$

By solving (1.35) for $\mathbf{a}_{i+1}$ we get

$$\mathbf{a}_{i+1} = (\mathbf{C} + \Delta t \Theta \mathbf{K})^{-1}\left[(\mathbf{C} - \Delta t \mathbf{K}(1 - \Theta))\mathbf{a}_i + \Delta t \bar{\mathbf{f}}\right] \tag{1.39}$$

If a proper choice of Θ is made the equation above may be solved for $\mathbf{a}_{i+1}$ when the current temperatures $\mathbf{a}_i$ are known. Equation (1.39) may also be written as

$$\boxed{\hat{\mathbf{K}}\mathbf{a}_{i+1} = \hat{\mathbf{f}}} \tag{1.40}$$

where

$$\boxed{\begin{aligned} \hat{\mathbf{K}} &= (\mathbf{C} + \Delta t \Theta \mathbf{K}) \\ \hat{\mathbf{f}} &= \left[(\mathbf{C} - \Delta t \mathbf{K}(1 - \Theta))\mathbf{a}_i + \Delta t \bar{\mathbf{f}}\right] \end{aligned}} \tag{1.41}$$

$\hat{\mathbf{K}}$ and $\hat{\mathbf{f}}$ only contain known values and may be calculated at step $\mathbf{a}_i$.

The final task is now to choose the weight parameter Θ. It was shown earlier that the weight function introduced in the weak form of the steady state heat flow could be chosen in several different manners. This is also the case for the weight parameter Θ.

1.6. Choice of weight parameter Θ

The point collocation method, described in Ottosen and Peterson, is adopted to determine the weight parameter Θ. The Dirac's δ-function may be chosen at different times in the time-step. Three common choices will be examined here; at $\tau = 0$, at $\tau = \Delta t/2$ and at $\tau = \Delta t$.

Choosing $w = \delta(\tau - 0)$

The weight parameter Θ was defined according to (1.34) that for a choice of w as a dirac's δ-function at $\tau = 0$ is written

$$\Theta = \frac{1}{\Delta t} \frac{\int_0^{\Delta t} \delta(\tau - 0)\tau \, d\tau}{\int_0^{\Delta t} \delta(\tau - 0) \, d\tau} = \frac{1}{\Delta t} \frac{0}{1} = 0 \tag{1.42}$$

This choice result in $\Theta = 0$ giving that the new temperature at $\mathbf{a}_{i+1}$ may, according to equation (1.39), be written as

$$\mathbf{a}_{i+1} = \mathbf{a}_i - \mathbf{C}^{-1}\Delta t \mathbf{K} \mathbf{a}_i + \mathbf{C}^{-1}\Delta t \bar{\mathbf{f}} \tag{1.43}$$

This choice result in a method that is called *Forward Euler* or *Explicit Euler* since the temperatures at the next time-step is calculated from the temperatures at the current time step only. Moreover, no inversion is required for the the conduction matrix $\mathbf{K}$.

Choosing $w = \delta(\tau - \Delta t)$

For a choice of w as a dirac's δ-function at $\tau = \Delta t$ equation (1.34) becomes

$$\Theta = \frac{1}{\Delta t} \frac{\int_0^{\Delta t} \delta(\tau - \Delta t)\tau \, d\tau}{\int_0^{\Delta t} \delta(\tau - \Delta t) \, d\tau} = \frac{1}{\Delta t} \frac{\Delta t}{1} = 1 \tag{1.44}$$

This choice result in $\Theta = 1$ giving that the new temperature at $\mathbf{a}_{i+1}$ may, according to equation (1.39), be written as

$$\mathbf{a}_{i+1} = (\mathbf{C} + \Delta t \mathbf{K})^{-1}(\mathbf{C}\mathbf{a}_i + \Delta t \bar{\mathbf{f}}) \tag{1.45}$$

This choice result in a method that is called *Backward Euler* and is an implicit method since it requires an inversion of the conduction matrix $\mathbf{K}$.

Choosing $w = \delta(\tau - \Delta t/2)$

For a choice of w as a dirac's δ-function at $\tau = \Delta t/2$ equation (1.34) becomes

$$\Theta = \frac{1}{\Delta t} \frac{\int_0^{\Delta t} \delta(\tau - \frac{\Delta t}{2})\tau \, d\tau}{\int_0^{\Delta t} \delta(\tau - \frac{\Delta t}{2}) \, d\tau} = \frac{1}{\Delta t} \frac{\Delta t/2}{1} = \frac{1}{2} \tag{1.46}$$

This choice result in $\Theta = 1/2$ giving that he new temperature at $\mathbf{a}_{i+1}$ may, according to equation (1.39), be written as

$$\mathbf{a}_{i+1} = (\mathbf{C} + \frac{\Delta t}{2}\mathbf{K})^{-1}\left[(\mathbf{C} - \frac{\Delta t}{2}\mathbf{K})\mathbf{a}_i + \Delta t\bar{\mathbf{f}}\right] \tag{1.47}$$

This choice result in a method that is called *Crank-Nicolson* from its founders and is also an implicit method.

The choice of $\Theta = 0$ is conditionally stable which means that a stable solution is only achieved for Δt smaller than a certain limit.

EXAMPLE 2 - Transient heat flow through a concrete wall
Consider a 1m thick concrete wall. The temperature distribution for a transient temperature change in the wall is going to be analysed for 12 hours of total time. One-dimensional flow is assumed since the analaysis is made for a part of the wall far from the edges of the wall. On the inside boundary the temperature is held constant at 0°C and on the outside boundary the temperature is changed from 0°C to 20°C at t=0. The initial temperature is 0°C. The density $\rho = 2400kg/m^3$, the specific heat capacitivity $c = 1000J/(K{\cdot}kg)$ and the thermal conductivity $k = 1.4J/(K{\cdot}ms)$. The calculation is made for $1m^2$ of the wall. The wall is divided with 10 one dimensional linear elements. The element stiffness matrices becomes

$$\mathbf{K}^e = \begin{bmatrix} 14 & -14 \\ -14 & 14 \end{bmatrix}$$

and the element capcitivity matrices

$$\mathbf{C}^e = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} 10^4$$

Analyses are made with the time stepping scheme described above for the three choices of $\Theta = 0$, 0.5 and 1 is performed. The total analysis time is $12 \cdot 3600$ s.

First, a choice of $\Delta t = 3600$ s is made. Thus analyses for 12 time steps are performed. Figure 1.2 show the temperature in the middle of the wall, $(x = 0.5)$ for the three choices of Θ and $\Delta t = 1 \cdot 3600$.

It is evident that the solution will become unstable for euler forward for a choice of Δt=3600s.

Secondly, a choice of $\Delta t = 0.24 \cdot 3600 = 864$ s is made. Thus analyses for 50 time steps are performed. Figure 1.3 show the temperature in the middle of the wall, $(x = 0.5)$ for the three choices of Θ and $\Delta t = 1 \cdot 3600$. Figure 1.3 show that the choice of $\Theta = 0.5$, for this example, gives the solution that is closest to the exact solution.

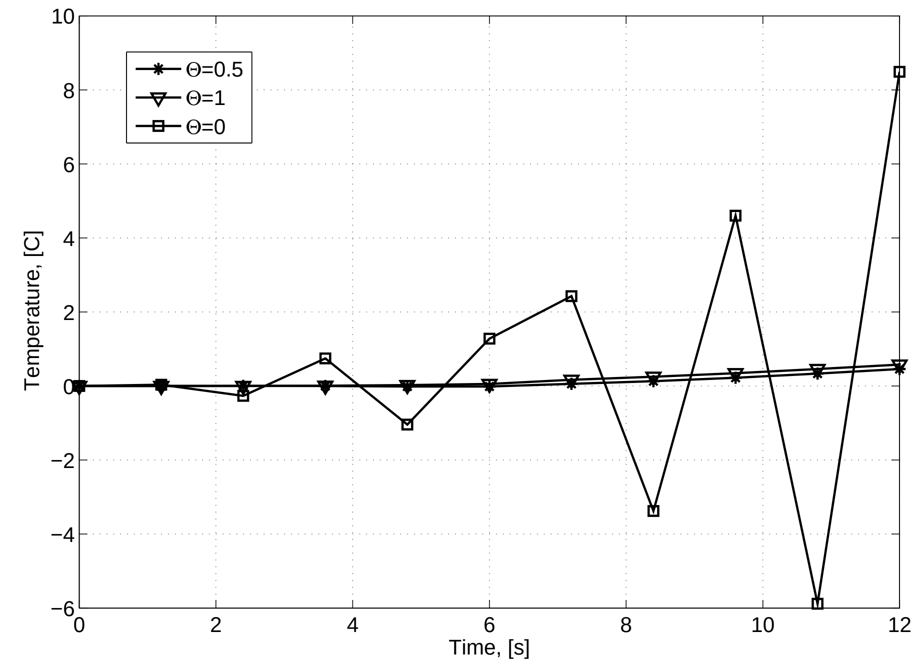


Figure 1.2: *Temperature at x=0 for the first 12 hours with a time step* Δt*=3600s.*

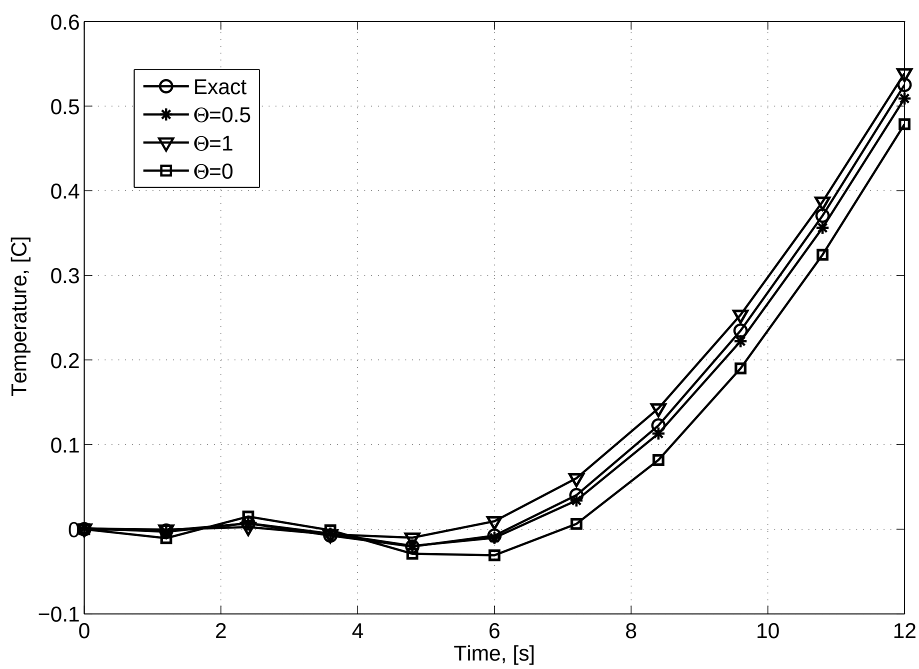


Figure 1.3: *Temperature at x=0 for the first 12 hours with a time step* Δt*=864s.*

2. Two- and three-dimensional transient heat flow

2.1. Two dimensional transient heat equation - strong form

As for one dimension the balance equation for the steady-state case may be rewritten to be valid for the transient case by adding a heat capacity term as

$$\boxed{-\operatorname{div}(t_h q) + t_h Q = \rho t_h c \frac{dT}{dt}} \tag{2.1}$$

where t_h is the thickness, q is the flux, ρ is the density of the materia and c is the heat capacity. By inserting the constitutive relation for two dimensional heat conduction, *Fourier's law*, that is written

$$\boxed{\mathbf{q} = -\mathbf{D}\nabla T} \tag{2.2}$$

The strong form of two-dimensional transient heat flow may now be formulated as

Strong form of two-dimensional transient heat flow

$$\begin{aligned} &\operatorname{div}(t_h \mathbf{D}\nabla T) + t_h Q = \rho t_h c \frac{dT}{dt}; \quad \text{in region } A \\ &q_n = h \quad \text{on } L_h \\ &T = g \quad \text{on } L_g \end{aligned} \tag{2.3}$$

2.2. Weak form of two-dimensional transient heat flow

The strong form of two-dimensional transient heat flow may be reformulated into a weak form in the same manner as for the steady state heat flow. The weak form is established by multiply the balance equation (2.1) with an arbitrary time-independent weight function $v(x, y)$ and integrating over the region.

$$-\int_A v \operatorname{div}(t_h \mathbf{q}) dA + \int_A v t_h Q da = \int_A v \rho t_h c \frac{dT}{dt} dA \tag{2.4}$$

The first term in this equation may be integrated by parts using the Green-Gauss theorem

$$\int_A v \operatorname{div}(t_h \mathbf{q}) dA = \oint_L v t_h \mathbf{q}^T \mathbf{n} dL - \int_A (\nabla v)^T t_h \mathbf{q} dA \tag{2.5}$$

Use of this expression in (2.4) implies that

$$\int_A (\nabla v)^T t_h \mathbf{q} dA = \oint_L v t_h \mathbf{q}^T \mathbf{n} dL - \int_A v t_h Q da + \int_A v \rho t_h c \frac{dT}{dt} dA \tag{2.6}$$

Now, using that $\mathbf{q} = -\mathbf{D}\nabla T$, inserting the natural boundary condition $q_n = h$ on L_h and rearranging the terms, the final weak form of two dimensional transient heat flow is established.

Weak form of two-dimensional transient heat flow

$$\int_A (\nabla v)^T t_h \mathbf{D} \nabla T dA + \int_A \rho t_h c \frac{dT}{dt} dA = -\oint_{L_h} v t_h h dL - \oint_{L_g} v t_h q_n dL + \int_A v t_h Q dA$$

$T = g$ on surface S_g

(2.7)

The main difference of the weak form of transient heat flow compared to the corresponding steady state case is that we now have derivatives both with respect to the spatial coordinates and to the time. This implies that approximations for the temperature must be made in both spatial coordinates and the time.

2.3. Spatial approximation of two-dimensional transient heat flow

As for the case of steady state heat flow an approximation of the temperature is introduced. The approximation may for the transient case be separated as

$$T(x, y, t) = \mathbf{N}(x, y)\mathbf{a}(t) \tag{2.8}$$

where $\mathbf{N}(x, y)$ are the shape functions that is functions of x and y whereas $\mathbf{a}(t)$ are the nodal temperatures that is functions of time. The weak form (2.7) contain both ∇T and dT/dt. Since $\mathbf{a}$ is independent of the spatial co-ordinates we may write

$$\nabla T = \nabla \mathbf{N}\mathbf{a} = \mathbf{B}\mathbf{a} \tag{2.9}$$

and since $\mathbf{N}$ is independent of t we may write

$$\frac{dT}{dt} = \mathbf{N}\frac{d\mathbf{a}}{dt} = \mathbf{N}\dot{\mathbf{a}} \tag{2.10}$$

Adopting the Galerkin method the scalar weight function v is chosen as

$$v = \mathbf{N}\mathbf{c} = \mathbf{c}^\mathbf{T}\mathbf{N}^\mathbf{T} \tag{2.11}$$

and

$$\nabla v = \mathbf{c^T B^T} \quad \text{where} \quad \mathbf{B^T} = \nabla \mathbf{N^T} \tag{2.12}$$

The approximation (1.10) and the choice of the weight function (1.13) is inserted in the weak form of the one dimensional transient heat flow (2.6) and also using the fact that the vector $\mathbf{c}$ is independent of x and y results in

$$\mathbf{c^T}\left(\int_A (\mathbf{B^T} t_h \mathbf{DB} dA \mathbf{a} + \int_A \mathbf{N^T} \rho t_h c \mathbf{N} dA\, \dot{\mathbf{a}} + \oint_{L_h} \mathbf{N^T} t_h h dL + \oint_{L_g} \mathbf{N^T} t_h q_n dL - \int_A \mathbf{N^T} t_h Q dA\right) = 0 \tag{2.13}$$

where the vector $\mathbf{a}$ now contain the nodal temperatures as function of the time and $\dot{\mathbf{a}}$ contain time derivatives of the nodal temperatures. As this expression is valid for arbitrary $\mathbf{c^T}$ vectors it is concluded that

$$\int_A (\mathbf{B^T} t_h \mathbf{DB} dA \mathbf{a} + \int_A \mathbf{N^T} \rho t_h c \mathbf{N} dA\, \dot{\mathbf{a}} = -\oint_{L_h} \mathbf{N^T} t_h h dL - \oint_{L_g} \mathbf{N^T} t_h q_n dL + \int_A \mathbf{N^T} t_h Q dA \tag{2.14}$$

which also may be written as

$$\boxed{\mathbf{Ka} + \mathbf{C}\dot{\mathbf{a}} = \mathbf{f_b} + \mathbf{f_l}} \tag{2.15}$$

where

$$\boxed{\begin{aligned} \mathbf{K} &= \int_A \mathbf{B^T} t_h \mathbf{D}\,\mathbf{B}\, dA \\ \mathbf{C} &= \int_A \mathbf{N^T} \rho t_h c \mathbf{N}\, dA \\ \mathbf{f_b} &= -\oint_{L_h} \mathbf{N^T} t_h h dL - \oint_{L_g} \mathbf{N^T} t_h q_n dL \\ \mathbf{f_l} &= \int_A \mathbf{N^T} t_h Q\, dA \end{aligned}} \tag{2.16}$$

Defining the force vector $\mathbf{f}$ as

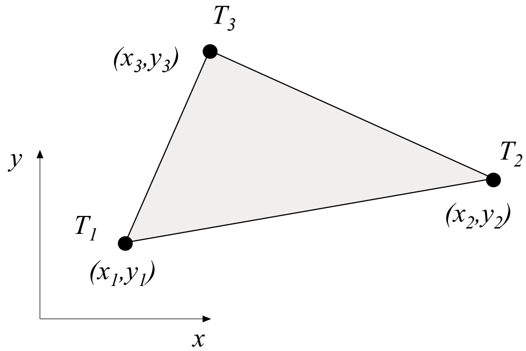
$$\mathbf{f} = \mathbf{f_b} + \mathbf{f_l} \tag{2.17}$$

equation 2.25 may be written in a compact form as

$$\boxed{\mathbf{Ka} + \mathbf{C}\dot{\mathbf{a}} = \mathbf{f}} \tag{2.18}$$

Since the vector $\mathbf{a}$ contain the nodal temperatures as function of the time and $\dot{\mathbf{a}}$ contain time derivatives of the nodal temperatures (1.17) represents a system of ODE's (ordinary differential equations) of order 1 and an approximation in time must also be made. The matrix $\mathbf{C}$ is called the *capacity* matrix and is the only term that differs from the steady state case.

EXAMPLE 2 - Capacity matrix $\mathbf{C}^e$ *for a two-dimensional linear triangle element*



The capacity matrix in two-dimensions is written

$$\mathbf{C} = \int_A \mathbf{N}^{\mathbf{T}} \rho t_h c \mathbf{N} \, dA \tag{2.19}$$

For a linear triangle element it becomes

$$\mathbf{C}^e = \rho t_h c A_e \begin{bmatrix} 1/6 & 1/12 & 1/12 \\ 1/12 & 1/6 & 1/12 \\ 1/12 & 1/12 & 1/6 \end{bmatrix} \tag{2.20}$$

where A_e is the element area.

2.4. Approximation in time of two-dimensional transient heat flow

The finite element equations now have to be approximated in time. Different choices of the time approximation may be made but as for the one-dimensional case, only a linear time approximation is considered. Since the approximation in time is scalar, the time approximation that was made for the one-dimensional case is also valid for two- and three-dimensional cases. It was concluded that the temperatures at the next time step $\mathbf{a}_{i+1}$ could be written

$$\boxed{\begin{aligned} \hat{\mathbf{K}} &= (\mathbf{C} + \Delta t \Theta \mathbf{K}) \\ \hat{\mathbf{f}} &= \left[(\mathbf{C} - \Delta t \mathbf{K}(1 - \Theta))\mathbf{a}_i + \Delta t \bar{\mathbf{f}} \right] \end{aligned}} \tag{2.21}$$

where the average force $\bar{\mathbf{f}}$ is

$$\bar{\mathbf{f}} = \mathbf{f}_i + \Theta(\mathbf{f}_{i+1} - \mathbf{f}_i) \tag{2.22}$$

$\hat{\mathbf{K}}$ and $\hat{\mathbf{f}}$ only contain known values that may be calculated at step $\mathbf{a}_i$. The choices of the weight parameter Θ that was made for the one-dimensional case is also valid for the two- and three-dimensional cases.

2.5. Three-dimensional transient heat flow

The strong form of three-dimensional form of transient heat flow may be directly stated as

Strong form of three-dimensional transient heat flow

$$
\begin{aligned}
&div(\mathbf{D}\nabla T) + Q = \rho c \frac{dT}{dt}; \quad \text{in region } V \\
&q_n = h \quad \text{on surface } S_h \\
&T = g \quad \text{on surface } S_g
\end{aligned}
\tag{2.23}
$$

and the corresponding weak form as

Weak form of three-dimensional transient heat flow

$$
\begin{aligned}
&\int_V (\nabla v)^T \mathbf{D} \nabla T dV + \int_V \rho c \frac{dT}{dt} dV = -\int_{S_h} v h dS - \int_{S_g} v q_n dS + \int_V v Q dV \\
&T = g \quad \text{on surface } S_g
\end{aligned}
\tag{2.24}
$$

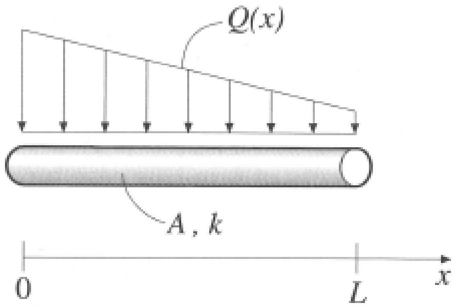
inserting approximation of $T(x, y, z, t)$ and the choice of the weight function $v(x, y, z)$ according to Galerkin's method the final fe-form yields

$$
\boxed{\mathbf{K}\mathbf{a} + \mathbf{C}\dot{\mathbf{a}} = \mathbf{f_b} + \mathbf{f_l}}
\tag{2.25}
$$

where

$$
\begin{aligned}
\mathbf{K} &= \int_V \mathbf{B^T} \mathbf{D}\, \mathbf{B}\, dV \\
\mathbf{C} &= \int_V \mathbf{N^T} \rho c \mathbf{N}\, dV \\
\mathbf{f_b} &= -\int_{S_h} v h dS - \int_{S_g} v q_n dS \\
\mathbf{f_l} &= \int_V \mathbf{N^T} Q\, dV
\end{aligned}
\tag{2.26}
$$

Exercises – Transient heat flow



T1.

Consider a bar of length L, constant cross-sectional area A and constant thermal conductivity k. The bar is made of steel with density ρ and heat capacity c and it is provided with an external heat supply $Q(x) = Q_0(10 - 6x/L)$ of dimension [W/m]. The flow of heat is assumed to be one-dimensional in the axial direction of the bar.

The balance equation for transient heat conduction is given by

$$-\frac{d}{dx}(Aq) + Q = \rho A c \frac{dT}{dt}$$

(a) Using the boundary conditions $T(0) = 80°C$ and $q(L) = 10^5$ W/m^2 for $t \geq 0$, give the strong formulation of the problem.
(b) Derive the corresponding weak formulation.
(c) Derive the FE formulation of the problem.
(d) Consider the bar as one element with linear temperature approximation and determine $\mathbf{K}$, $\mathbf{C}$, $\mathbf{f_b}$ and $\mathbf{f_l}$.
(e) Divide the bar into three equally long elements with linear temperature approximation in each of the elements and determine $\mathbf{K}$, $\mathbf{C}$, $\mathbf{f_b}$ and $\mathbf{f_l}$.

At $t = 0$, the temperature is $80°C$ in the whole bar. To determine the temperature distribution in the bar for $t > 0$, a time-stepping procedure is adopted. A linear approximation of the temperature variation between time t_i and t_{i+1} is assumed, yielding the time-stepping scheme

$$\hat{\mathbf{K}}\mathbf{a}_{i+1} = \hat{\mathbf{f}}$$

where

$$\hat{\mathbf{K}} = (\mathbf{C} + \Delta t \Theta \mathbf{K})$$

$$\hat{\mathbf{f}} = \left[\left(\mathbf{C} - \Delta t \mathbf{K}(1 - \Theta)\right)\mathbf{a}_i + \Delta t \bar{\mathbf{f}}\right]$$

The Forward Euler method is given by $\Theta = 0$ and the Backward Euler method given by $\Theta = 1$.

(f) Consider the equation system obtained in (d). For $L = 6$ m, $A = 2 \cdot 10^{-3}$ m^2, $k = 10^4$ W/mK, $Q_0 = 9$ W/m, $\rho = 7800$ kg/m^3 and $c = 490$ J/K·kg, determine $\mathbf{a}_1$ manually by calculating $\hat{\mathbf{K}}$ and $\hat{\mathbf{f}}$ with use of $\mathbf{a}_0$. Insert the boundary condition ($T(0) = 80°C$) and solve for $\mathbf{a}_1$. Employ both the Forward Euler method and Backward Euler method, using the time step $\Delta t = 120$ s, to solve the problem.

(g) With help of Matlab, plot $T(L, t)$ for $0 \leq t \leq 10$ h, using again both the Forward Euler method and Backward Euler method, with the time step $\Delta t = 120$ s, to solve the problem. What is the value at $T(L, t = 10h)$? Compare with exercise 9.1 (g).

(h) Repeat (g), now using the equation system obtained in (e).

(i) Repeat (g) and (h) using $\Delta t = 360$ s, what is observed for the different time stepping methods?

Solutions, T1

(a)

$$Ak\frac{d^2T}{dx^2} + Q = \rho Ac\frac{dT}{dt}$$

$$T(x=0) = 80°C$$

$$q(x=L) = -\left(k\frac{dT}{dx}\right)_{x=L} = 10^5\ W/m^2$$

(b)

$$\int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}dx + \int_0^L v\rho Ac\frac{dT}{dt}dx = -(vA\cdot 10)_{x=L} + (vAq)_{x=0} + \int_0^L vQdx$$

$$T(x=0) = 80°C$$

(c)

$$\mathbf{Ka} + \mathbf{C\dot{a}} = \mathbf{f}_\mathrm{b} + \mathbf{f}_\mathrm{l}$$

$$\mathbf{K} = \int_0^L \mathbf{B}^\mathrm{T}Ak\mathbf{B}dx\ ; \qquad \mathbf{C} = \int_0^L \mathbf{N}^\mathrm{T}\rho Ac\mathbf{N}dx$$

$$\mathbf{f}_\mathrm{b} = -[\mathbf{N}^\mathrm{T}Aq]_0^L\ ; \qquad \mathbf{f}_\mathrm{l} = \int_0^L \mathbf{N}^\mathrm{T}Qdx$$

(d)

$$\mathbf{K} = \frac{Ak}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \mathrm{W/K}; \qquad \mathbf{C} = \frac{\rho AcL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\ \mathrm{J/K}$$

$$\mathbf{f_b} = \begin{bmatrix} (Aq)_{x=0} \\ -A\cdot 10 \end{bmatrix}\ \mathrm{W}; \qquad \mathbf{f_l} = \mathrm{Q_0L}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\ \mathrm{W}$$

(e)

$$\mathbf{K} = \frac{Ak}{(L/3)}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\ \mathrm{W/K}; \qquad \mathbf{C} = \frac{\rho AcL}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\ \mathrm{J/K}$$

$$\mathbf{f_b} = \begin{bmatrix} (Aq)_{x=0} \\ 0 \\ 0 \\ -A\cdot 10 \end{bmatrix}\ \mathrm{W}; \qquad \mathbf{f_l} = \mathrm{Q_0L}\begin{bmatrix} 1.556 \\ 2.667 \\ 2.000 \\ 0.778 \end{bmatrix}\ \mathrm{W}$$

(f)

$$\mathbf{a}_0 = \begin{bmatrix} 80 \\ 80 \end{bmatrix} \quad \Rightarrow \quad \text{Forward Euler: } \hat{\mathbf{K}} = \begin{bmatrix} 15288 & 7644 \\ 7644 & 15288 \end{bmatrix} \text{J/K}; \quad \hat{\mathbf{f}} = \begin{bmatrix} 1860480 + \Delta t A q_{(x=0)} \\ 1830000 \end{bmatrix} \text{J}$$

$$\text{Backward Euler: } \hat{\mathbf{K}} = \begin{bmatrix} 15688 & 7244 \\ 7244 & 15688 \end{bmatrix} \text{J/K}; \quad \hat{\mathbf{f}} = \begin{bmatrix} 1860480 + \Delta t A q_{(x=0)} \\ 1830000 \end{bmatrix} \text{J}$$

Solving the resulting systems:

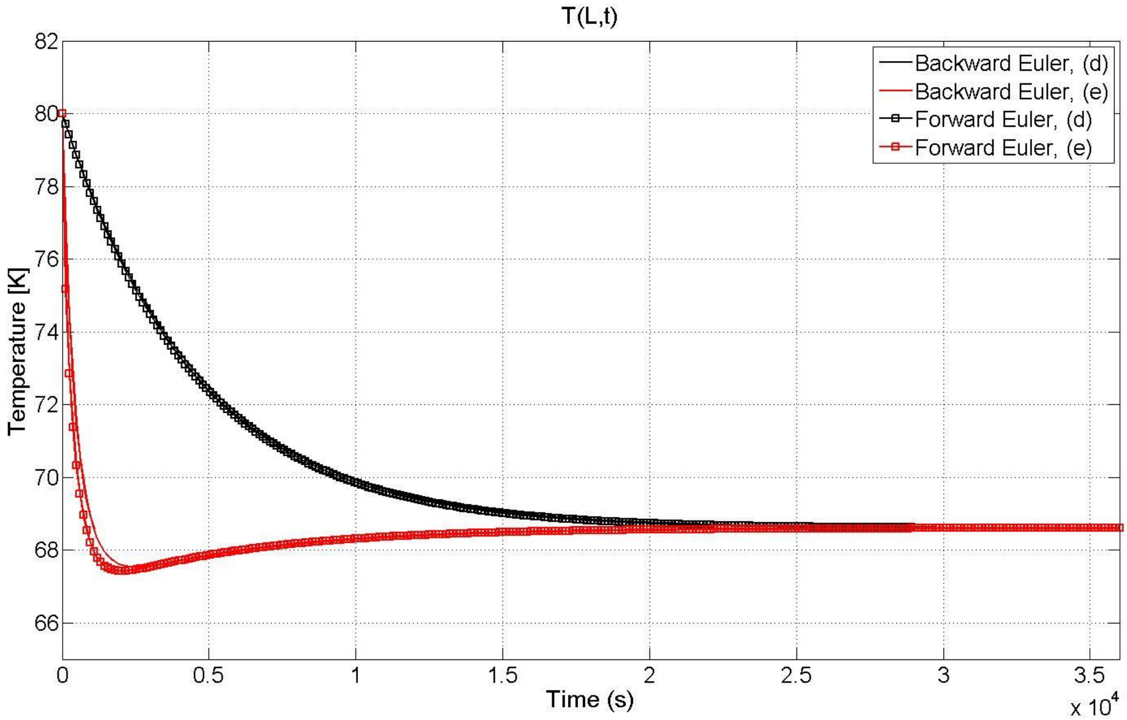
Forward Euler: $$\begin{bmatrix} 15288 & 7644 \\ 7644 & 15288 \end{bmatrix} \begin{bmatrix} 80 \\ T_{2,1} \end{bmatrix} = \begin{bmatrix} 1860480 + \Delta t A q_{(x=0)} \\ 1830000 \end{bmatrix}$$

$$\Rightarrow \quad 7644 \cdot 80 + 15288 \cdot T_{2,1} = 1830000 \quad \Rightarrow \quad \mathbf{a}_1 = \begin{bmatrix} 80 \\ 79.70 \end{bmatrix} \text{K}$$

Backward Euler: $$\begin{bmatrix} 15688 & 7244 \\ 7244 & 15688 \end{bmatrix} \begin{bmatrix} 80 \\ T_{2,1} \end{bmatrix} = \begin{bmatrix} 1860480 + \Delta t A q_{(x=0)} \\ 1830000 \end{bmatrix}$$

$$\Rightarrow \quad 7244 \cdot 80 + 15688 \cdot T_{2,1} = 1830000 \quad \Rightarrow \quad \mathbf{a}_1 = \begin{bmatrix} 80 \\ 79.71 \end{bmatrix} \text{K}$$

(g,h)



$T(L, t = 10h) = 68.6°C$ (in all four cases). The same value as obtained in 9.1 (g), where a steady state was assumed. Hence, at $t = 10h$ we have reached a steady state.