Matrix Algebra

Matrix Algebra

- Definitions
- Addition and Subtraction
- Multiplication
- Determinant
- Inverse
- System of Linear Equations
- Quadratic Forms
- Partitioning
- Differentiation and Integration

2.1 Definitions

- Scalar (0D matrix)

$$c$$

- Vector (1D matrix)

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Transpose

$$\mathbf{a}^{\mathrm{T}} = [a_1 \quad a_2 \quad a_3]$$

- Matrix (2D matrix)

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \\ C_{41} & C_{42} \end{bmatrix}$$

Transpose

$$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}$$

2.1 Definitions, contd

- Matrix size, [m x n] or [i x j], ([row x col])
  – Vector, [m x 1]

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \\ C_{41} & C_{42} \end{bmatrix}$$

[m x n] = [4 x 2]

- Square matrix, (m=n) [m x m]

- Symmetric matrix, $\mathbf{B}=\mathbf{B}^{\mathbf{T}}$

2.1 Definitions, contd

- Diagonal matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

- Identity matrix, (unit matrix), **AI=A**

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Zero matrix

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.2 Addition and Subtraction

- Vector

$$\mathbf{c} = \mathbf{a} \pm \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \pm \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}$$

- Matrix

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \pm \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} A_{11} \pm B_{11} & A_{12} \pm B_{12} \\ A_{21} \pm B_{21} & A_{22} \pm B_{22} \\ A_{31} \pm B_{31} & A_{32} \pm B_{32} \end{bmatrix}$$

2.3 Multiplication - scalar

- Scalar – vector multiplication

$$c\mathbf{a} = c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$$

- Scalar – matrix multiplication

$$c\mathbf{A} = c\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} cA_{11} & cA_{12} \\ cA_{21} & cA_{22} \\ cA_{31} & cA_{32} \end{bmatrix}$$

2.3 Multiplication - vector

- Scalar product, (vector – vector multiplication)

$$\mathbf{a}^T\mathbf{b} = [a_1 \quad a_2 \quad a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

[1 x n] [n x 1] = [1 x 1]

- Length of vector (cf. Pythagoras' theorem)

$$|\mathbf{a}| = (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}$$

$$\boxed{|\mathbf{a}| = (\mathbf{a}^T\mathbf{a})^{1/2}}$$

2.3 Multiplication - vector

- Matrix product

$$\mathbf{ab}^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

[m x 1] [1 x n] = [m x n]

2.3 Multiplication - matrix

- Matrix – vector multiplication

$$\mathbf{c} = \mathbf{A}\mathbf{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \\ A_{31}x_1 + A_{32}x_2 \end{bmatrix}$$

[m x n] [n x 1] = [m x 1]

[3 x 2] [2 x 1] = [3 x 1]

- Vector – matrix multiplication

$$\mathbf{c}^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{A} = [x_1 \quad x_2] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

$$= [x_1 A_{11} + x_2 A_{21} \quad x_1 A_{12} + x_2 A_{22} \quad x_1 A_{13} + x_2 A_{23}]$$

[1 x m] [m x n] = [1 x n]

2.3 Multiplication - matrix

- Matrix – matrix multiplication

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{bmatrix}$$

[m x n] [n x p] = [m x p]

Note! $\mathbf{AB} \neq \mathbf{BA}$

2.3 Multiplication – matrix, contd

- Product of transposed matrices

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

$$(\mathbf{ABC})^{\mathrm{T}} = ((\mathbf{AB})\mathbf{C})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}(\mathbf{AB})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

- Distribution law

$$(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{Ax} + \mathbf{Bx}$$

$$\mathbf{x}^{\mathrm{T}}(\mathbf{A} + \mathbf{B}) = \mathbf{x}^{\mathrm{T}}\mathbf{A} + \mathbf{x}^{\mathrm{T}}\mathbf{B}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$

2.4 Determinant

- The determinant may be calculated for any square matrix, [n x n]
- Cofactor of matrix, **A** *(i=row, k=column)*

$$A^{c}_{ik} = (-1)^{i+k} \det M_{ik}$$

- Expansion formula

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{ik} A^{c}_{ik}$$

where i indicates any row number in the range $1 \le i \le n$.

2.4 Determinant, examples

Cofactors

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{A}_{11}^{c} = (-1)^{(1+1)} 4 = 4$$

$$\mathbf{A}_{12}^{c} = (-1)^{(1+2)} 2 = -2$$

Determinant

$$\det \mathbf{A} = 1 \cdot 4 + 2(-4) = -2$$

Cofactors

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 5 \end{bmatrix}$$

$$\mathbf{B}_{11}^{c} = (-1)^{(1+1)} (5 \cdot 5 - 6 \cdot 8) = -23$$

$$\mathbf{B}_{21}^{c} = (-1)^{(2+1)} (2 \cdot 5 - 3 \cdot 8) = -14$$

$$\mathbf{B}_{31}^{c} = (-1)^{(3+1)} (2 \cdot 6 - 3 \cdot 5) = -3$$

Determinant

$$\det \mathbf{B} = 1(-23) + 4 \cdot 14 + 7(-3) = 12$$

2.5 Inverse Matrix

- The inverse $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$ is defined by

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

- The inverse may be determined by the cofactors, where

$$\text{adj } \mathbf{A} = \begin{bmatrix} A_{11}^{c} & A_{12}^{c} & \cdots & A_{1n}^{c} \\ A_{21}^{c} & A_{22}^{c} & \cdots & A_{2n}^{c} \\ \vdots & \vdots & & \vdots \\ A_{n1}^{c} & A_{n2}^{c} & \cdots & A_{nn}^{c} \end{bmatrix}^{\mathrm{T}}$$

is the adjoint of $\mathbf{A}$ and

$$\boxed{\mathbf{A}^{-1} = \text{adj } \mathbf{A}/\det \mathbf{A}}$$

note that $\mathbf{A}^{-1}$ only exists if $\det \mathbf{A} \neq 0$

2.4 Inverse, examples

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{adj}\mathbf{A} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \text{adj}\mathbf{A} / \det \mathbf{A} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \frac{1}{(-2)}$$

What happens if $\det \mathbf{A} = 0$?

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 5 \end{bmatrix} \qquad \text{adj}\mathbf{B} = \begin{bmatrix} -23 & 22 & -3 \\ -14 & -16 & 6 \\ -3 & 6 & -3 \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{B}^{-1} = \text{adj}\mathbf{B} / \det \mathbf{B} = \begin{bmatrix} -23 & 14 & -3 \\ -22 & -16 & 6 \\ -3 & 6 & -3 \end{bmatrix} \frac{1}{12}$$

2.6 Systems of Linear Equations

-number of equations is equal to number of unknowns

- Linear equation system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

A is a square matrix [n x n],

x and **b** [n x 1] vector

$\mathbf{b} = \mathbf{0}$ => Homogeneous system

$\mathbf{b} \neq \mathbf{0}$ => Inhomogeneous system

- Assume that $\mathbf{det(A)} \neq \mathbf{0}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

2.6 Linear Equations

-number of equations is equal to number of unknowns

- Homogeneous system $\mathbf{b} = \mathbf{0}$, (*trivial solution: $x = 0$*)

$\mathbf{Ax} = \mathbf{0}$

- If det $\mathbf{A} = 0$, a non-trivial solution exists — Eigenvalue problems
- If det $\mathbf{A} \neq 0$, no non-trivial solution exists

- Inhomogeneous system $\mathbf{b} \neq \mathbf{0}$

$\mathbf{Ax} = \mathbf{b}; \quad \mathbf{b} \neq \mathbf{0}$

- If det $\mathbf{A} \neq 0$, one unique solution given by (2.53) exists
- If det $\mathbf{A} = 0$, no unique solution exists. Depending on the specific $\mathbf{b}$-matrix we may have no solution or an infinity of solutions

2.8 Quadratic forms and positive definiteness

- If

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}$$

- then the matrix $\mathbf{A}$ is positive definite

$$\text{If } \mathbf{A} \text{ is positive definite then } \det \mathbf{A} \neq 0$$

- If $\mathbf{A}$ is positive definite, all diagonal elements must be positive

2.9 Partitioning

The matrix $\mathbf{A}$ may be partitioned as

$$\mathbf{A} = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right]$$

and if

$$\mathbf{B} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix}$$

$$\mathbf{D} = [A_{31} \quad A_{32}]; \quad \mathbf{E} = [A_{33}]$$

$\mathbf{A}$ may be written

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$$

2.9 Partitioning, contd

An equation system can be written

$$\mathbf{Ax} = \mathbf{f}$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

introduce

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \mathbf{z} = [x_3]; \quad \mathbf{g} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}; \quad \mathbf{h} = [f_3]$$

The partitioned equation system is written

$$\begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}$$

or

$$\begin{aligned} \mathbf{By} + \mathbf{Cz} &= \mathbf{g} \\ \mathbf{Dy} + \mathbf{Ez} &= \mathbf{h} \end{aligned}$$

2.10 Differentiation and integration

- A matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

- Differentiation

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x} = \begin{bmatrix} \frac{\mathrm{d}A_{11}}{\mathrm{d}x} & \frac{\mathrm{d}A_{12}}{\mathrm{d}x} & \frac{\mathrm{d}A_{13}}{\mathrm{d}x} \\ \frac{\mathrm{d}A_{21}}{\mathrm{d}x} & \frac{\mathrm{d}A_{22}}{\mathrm{d}x} & \frac{\mathrm{d}A_{23}}{\mathrm{d}x} \end{bmatrix}$$

- Integration

$$\int \mathbf{A}\,\mathrm{d}x = \begin{bmatrix} \int A_{11}\,\mathrm{d}x & \int A_{12}\,\mathrm{d}x & \int A_{13}\,\mathrm{d}x \\ \int A_{21}\,\mathrm{d}x & \int A_{22}\,\mathrm{d}x & \int A_{23}\,\mathrm{d}x \end{bmatrix}$$