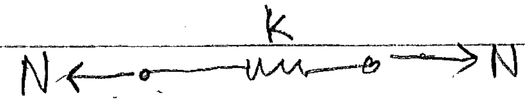
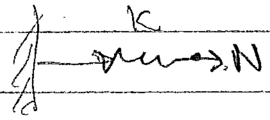
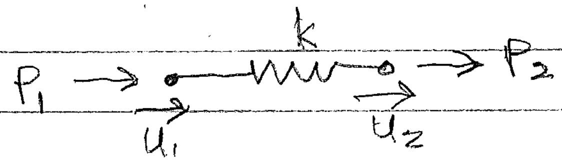
1/a)





$N = k \cdot \delta u$



$$N = k(u_2 - u_1)$$

$$\begin{cases} P_1 = -k(u_2 - u_1) \\ P_2 = k(u_2 - u_1) \end{cases} \xrightarrow{\text{on matrix form}} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

b)

Topology

el	dof	
1	2	3
2	2	4
3	3	4
4	1	4

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} k_4 & 0 & 0 & -k_4 \\ 0 & k_1+k_2 & -k_1 & -k_2 \\ 0 & -k_1 & k_1+k_3 & -k_3 \\ -k_4 & -k_2 & -k_3 & k_2+k_3+k_4 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

c)

$$k \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 3 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \Rightarrow \mathbb{K}_r = k \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$u_4 = u_0 \quad F_3 = 0 \qquad \det(\mathbb{K}_r) = 7k^2$

d) use now that $u_1 = u_2 = 0$, $F_3 = 0$ and $u_4 = u_0$

$$\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} u_3 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ F_4 \end{bmatrix}$$

$$2u_3 - u_0 = 0 \Rightarrow u_3 = \frac{u_0}{2}$$

$N^e = k(u_2^e - u_1^e)$

$N_1 = k(u_3 - u_2) = \frac{u_0}{2} k$

$N_2 = 2k(u_4 - u_2) = 2u_0 k$

$N_3 =$ [illegible] $\quad N_4 = u_0 k$

2/a

$$\frac{d}{dx}(Aq) = Q$$

$$\begin{cases} \frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = 0 & 0 \leq x \leq L \\ q = h \quad \text{at } x = 0 & \\ T = g \quad \text{at } x = L & \end{cases}$$

b/ natural, written in the weak form

essential, needs to be prescribed at at least one location

c/ multiply with $v(x)$ and integrate over region:

$$\int_0^L v\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) \cancel{dx} \int_0^L vQ\,dx = 0$$

Partial integration of first term:

$$\int_0^L v\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)dx = \left[vAk\frac{dT}{dx}\right]_0^L - \int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}dx$$

insert and use constitutive eq.

$$\begin{cases} \int_0^L \frac{dv}{dx}Ak\frac{dT}{dx}dx = -(vAq)_{x=L} + (vAh)_{x=0} + \int_0^L vQdx \\ T(x=L) = g \end{cases}$$

weak form

d) Approximation

$$T = N a \quad ; \quad \frac{dT}{dx} = B a \quad ; \quad B = \frac{dN}{dx}$$

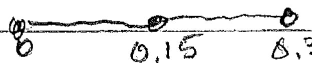
Galerkin

$$v = N c = c^T N^T \qquad \frac{dv}{dx} = c^T B^T$$

$$c^T \left(\int_0^L B^T A k B \, dx \; a \right) = c^T \left(-\left(N^T A q\right)_{x=L} + \left(N^T A h\right)_{x=0} + \int_0^L N^T Q \, dx \right)$$

e) one element, three nodes, $A = 1\,m^2$

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \qquad T(x) = \bar{N} \cdot \alpha = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$



$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0{,}15 & 0{,}15^2 \\ 1 & 0{,}3 & 0{,}3^2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad ; \quad a^e = C \alpha \Rightarrow \alpha = C^{-1} \cdot a^e$$

$$T(x) = \bar{N} C^{-1} \cdot a = N \cdot a$$

$$B = \frac{d\bar{N}}{dx} C^{-1} \quad ; \quad B^T = \left(C^{-1}\right)^T \frac{d\bar{N}^T}{dx}$$

$$K = \left(C^{-1}\right)^T \int_0^L \frac{d\bar{N}^T}{dx} k \frac{d\bar{N}}{dx} dx \; C^{-1},$$

$$f_L = \left(C^{-1}\right)^T \int_0^L \bar{N}^T Q \, dx \quad ; \quad f_b = \begin{bmatrix} h \\ 0 \\ -q \end{bmatrix}$$

$$\frac{d\bar{N}}{dx} = [0 \;\; 1 \;\; 2x]$$

e) forts

$$\mathbb{K} = (\mathbb{C}^{-1})^T k \int_0^L \begin{bmatrix} 0 \\ 1 \\ 2x \end{bmatrix} [0 \;\; 1 \;\; 2x] \, dx \, \mathbb{C}_1^{-1} =$$

$$= (\mathbb{C}^{-1})^T k \int_0^L \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2x \\ 0 & 2x & 4x^2 \end{bmatrix} dx \, \mathbb{C}^{-1} =$$

$$= (\mathbb{C}^{-1})^T k \begin{bmatrix} c & c & c \\ c & x & x^2 \\ c & x^2 & \frac{4}{3}x^3 \end{bmatrix}_0^L \mathbb{C}^{-1} = (\mathbb{C}^{-1})^T k \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & L^2 \\ 0 & L^2 & \frac{4}{3}L^3 \end{bmatrix} \mathbb{C}^{-1}$$

$k = 2{,}0 \text{ W/Km}$

$$\mathbb{K} = \frac{1}{9} \begin{bmatrix} 140 & -160 & 20 \\ -160 & 320 & -160 \\ 20 & -160 & 140 \end{bmatrix}$$

$$f_L = (\mathbb{C}^{-1})^T Q \int_0^L \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} dx = Q \cdot (\mathbb{C}_1^{-1})^T \begin{bmatrix} x \\ \frac{x^2}{2} \\ \frac{x^3}{3} \end{bmatrix}_0^L = Q (\mathbb{C}^{-1})^T \begin{bmatrix} L \\ L^2/2 \\ L^3/3 \end{bmatrix} = Q \cdot \begin{bmatrix} 0{,}05 \\ 0{,}2 \\ 0{,}05 \end{bmatrix}$$

f)

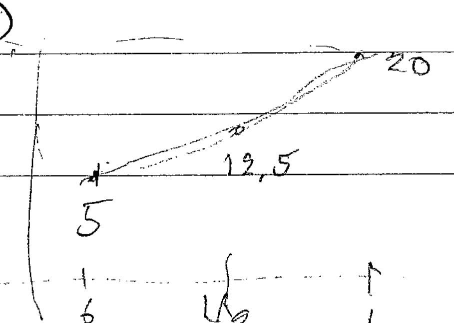
$$\frac{1}{9} \begin{bmatrix} 140 & -160 & 20 \\ -160 & 320 & -160 \\ 20 & -160 & 140 \end{bmatrix} \begin{bmatrix} 20 \\ T_2 \\ 5 \end{bmatrix} = \begin{bmatrix} f_{b1} \\ 0 \\ f_{b2} \end{bmatrix}$$

second row

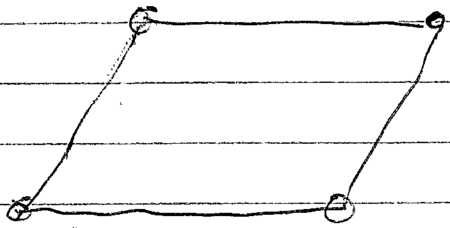
$$\cancel{\frac{1}{9}}\left(-160 \cdot 20 + 320 T_2 - 160 \cdot 5\right) = 0$$

$$T_2 = \frac{4000}{320} = 12{,}5$$

insert T_2 and mult. to determine f_{b1} and f_{b2}



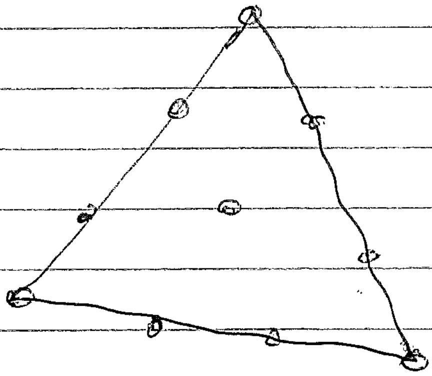
3a)



4-nodes

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

Parasitic



10-nodes

$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 x y^2 + \alpha_{10} y^3$$

no parasitic terms

b) completeness + compatibility = convergence

Both elements: Completeness $T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$ + other terms

Fulfilled!

Compatibility:

4 node $\quad T(x, ax+b) = \alpha_1 + \alpha_2 x + \alpha_3 (ax+b) + \alpha_4 x(ax+b)$

$$= \beta_1 + \beta_2 x + \beta_3 x^2$$

three terms but only two nodes! not ok!

10-node

$$T(x, ax+b) = \alpha_1 + \alpha_2 x + \alpha_3 (ax+b) + \alpha_4 x(ax+b) + \alpha_5 x(ax+b) + \alpha_6 (ax+b)^2 + \alpha_7 x^3 + \alpha_8 x^2 (ax+b) + \alpha_9 x (ax+b)^2 + \alpha_{10} (ax+b)^3 = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^4$$

4 terms and 4 nodes along line

$\Rightarrow$ OK

4a) <u>Approximation:</u>

$$T(x,y) = \mathbf{N}\mathbf{a} \quad ; \quad \nabla T = \nabla \mathbf{N}\mathbf{a} = \mathbf{B}\mathbf{a}$$

<u>Galerkin:</u> $v = \mathbf{N}\mathbf{c} = \mathbf{c}^T\mathbf{N}^T \qquad \nabla v = \mathbf{c}^T\nabla\mathbf{N}^T = \mathbf{c}^T\mathbf{B}$

insert in weak form:

$$\cancel{\mathbf{c}^T}\int_A \mathbf{B}^T\mathbf{D}t\mathbf{B}\,dA\,\mathbf{a} = \cancel{\mathbf{c}^T}\left(-\int_{\mathcal{L}_h}\mathbf{N}^T h t\,d\mathcal{L} - \int_{\mathcal{L}_g}\mathbf{N}^T q_n t\,d\mathcal{L} + \int_A \mathbf{N}^T Q t\,dA\right)$$

b) see Calfem manual: flw2te.m

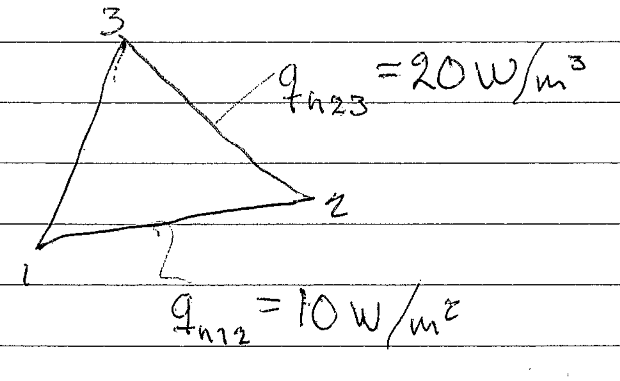
$$\mathbf{K}^e = (\mathbf{C}^{-1})^T\int_A \bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}t\,dA\cdot\mathbf{C}^{-1} = (\mathbf{C}^{-1})^T\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}t\,\mathbf{C}^{-1}\underbrace{\int_A dA}_{A_e =}$$

$$\mathbf{C} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \begin{aligned} A_e &= \\ &= \tfrac{1}{2}\det(\mathbf{C}) \\ &= 5/2 \end{aligned}$$

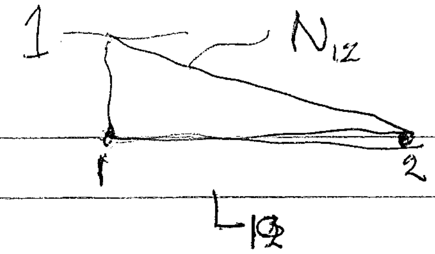
$$\bar{\mathbf{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{K} = 0{,}1\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

c) $$\mathbf{f}_l = \int_A \mathbf{N}^T Q t\,dA = Qt\int_A \mathbf{N}^T dA = \frac{Q\cdot A_e t}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d)



$$\mathbf{f}_b = \begin{bmatrix} \int_{\mathcal{L}_{12}} N_1^T q_{n12} t\,d\mathcal{L} + \int_{\mathcal{L}_{13}} N_1^T q_{n13} t\,d\mathcal{L} \\ \int_{\mathcal{L}_{12}} N_2^T q_{n12} t\,d\mathcal{L} + \int_{\mathcal{L}_{23}} N_2^T q_{n23} t\,d\mathcal{L} \\ \int_{\mathcal{L}_{13}} N_3^T q_{n13} t\,d\mathcal{L} + \int_{\mathcal{L}_{23}} N_3^T q_{n23} t\,d\mathcal{L} \end{bmatrix}$$



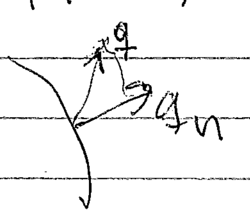
$$\int_{L_{12}} N_1 \, dL = \frac{L_{12}}{2}$$

$$f_b = \begin{bmatrix} 0,1 \cdot 10 \cdot \frac{L_{12}}{2} + q_{n13} \, 0,1 \, \frac{L_{13}}{2} \\ 0,1 \cdot 10 \, \frac{L_{12}}{2} + 0,1 \cdot 20 \cdot \frac{L_{23}}{2} \\ 0,1 \cdot q_{n13} \, \frac{L_{13}}{2} + 0,1 \cdot 20 \cdot \frac{L_{23}}{2} \end{bmatrix}$$

e) $\mathbb{K} \cdot a = \mathbb{f}$; three equation, three unknowns

T_2, T_3, q_{n13}

5/a) $q_n = q^T n$; positive if heat is leaving body



q_n [W/m²]

b) $$\int_A Q \, t \, dA = \oint_L q^T n t \, dL$$

c) see the book.