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How to derive an analytical network mechanics theory

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Abstract

This report was originally written as a part of the course material in the FPIRC course Paper Mechanics given at STFI/KTH. The aim of the report is to explain the ideas behind analytical network theories based on the homogeneous strain assumption in a simple way. One example, corresponding to the Cox model, [2], is also worked out in detail in the report.

1 Introduction

Many network theories rely on the assumption of *homogeneous strain.* This means that the strain is equal everywhere in a sheet, and thus equal to the average strain. In a heterogeneous material there is generally not a state of homogeneous strain since areas of less stiffness elongates more than stiffer areas when a sheet is subjected to extension. Homogeneous strain is a better approximation the more homogeneous the material. In the following section it is outlined how a homogeneous strain theory can be formulated. Thereafter a simple example is shown in detail.

2 Outline of method

1) Basic assumptions:
Assume a 2-dimensional fibre network of straight identical fibres positioned uniformly in the sheet. Fibre properties could also be assumed to follow some statistical distribution.

2) Homogeneous strain assumption:
Each bond center is assumed to be displaced exactly as the corresponding point would have been if the material had been homogeneous. The part of a fibre between two neighboring bond centres is denoted a fibre segment. Some assumption could also be made regarding the rotation of the fibre segment end points, e.g. no rotation or rotation determined by the strain field.

3) Calculate the displacement and rotation of the fibre segment end points as a function of fibre orientation, θ, and global strain, $(\epsilon_x, \epsilon_y, \gamma_{xy})$.

4) Make assumption regarding what types of deformation the fibre can sustain, eg axial elongation, bending or shear. Assume constitutive behaviour of the fibre, e.g. linear elastic. Assume behaviour in relative rotation of fibres at bonds, e.g. free relative rotation, moment in bond proportional to relative rotation or rigid bonds. These assumptions must be consistent with the ones in 2).

5) Calculate the forces in the fibre segment end points as a function of θ and $(\epsilon_x, \epsilon_y, \gamma_{xy})$ by using theory for structural elements like bars or beams. If no load is applied to the fibre between bonds the axial and shear force is constant along the fibre segment, and the bending moment varies linearly.

6) Determine the number of fibres, of a certain angle θ, crossing a line of unit length parallel to the x-axis and y-axis respectively. This is a function of the total fibre length per unit area, ρ, and the orientation distribution, $f(\theta)$.

7) Calculate the total force per unit length·thickness in the x-direction on a line parallel to the y-axis, σ_x, in the y-direction on a line parallel to the y-axis, τ_{xy}, and in the in the y-direction on a line parallel to the x-axis, σ_y. An alternative method is to calculate the elastic energy per unit volume of the system, W, and obtaining σ_x as $\dfrac{\partial W}{\partial \epsilon_x}$ etc.
8) Determine homogenized constitutive parameters. From 7) we have a relation between stress (force per unit area) and strain. By assuming plane stress we have the same relation for a homogeneous material. By identification of coefficients the elastic parameters, e.g. for an isotropic material E and ν, can be determined.

3 Example of calculations

1) Assume a 2-dimensional fibre network of straight identical fibres positioned uniformly in the sheet. Fibre length is denoted l_f and cross-section area is denoted A_f.

2) Each bond center is assumed to be displaced exactly as the corresponding point would have been if the material had been homogeneous. The rotation of the fibre segment ends is according to the strain field. This means that the fibre remains straight after deformation and there is no use trying to incorporate bending in the model.

3) From a textbook in solid mechanics, [1], we have the formula for transformation of strain, stating the axial strain in an element oriented in an angle θ to the x-axis, when the global strain is $(\epsilon_x, \epsilon_y, \gamma_{xy})$.

$$\epsilon'_x = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta \tag{1}$$

4) The fibres are linear elastic with Young's modulus E_f, and support only axial force, that is, act like bars. The rotation of a fibre due to global deformation of the sheet is dependent on fibre orientation. Because of this there is relative rotation between the fibres at a bond where two fibres of different orientation cross each other. It is assumed that there is no resistance to relative rotation of the fibres at the bonds.

5) From Hooke's law for a bar ($\sigma = E\epsilon$ or $\dfrac{F}{A} = E\epsilon$) we get the force, F_f, in a fibre subjected to axial strain ϵ_a

$$F_f = E_f A_f \epsilon_a = E_f A_f(\epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta) \tag{2}$$

6) The orientation distribution of the fibres is uniform, that is $f(\theta) = \dfrac{1}{\pi}$. The number of fibres at angle θ which intersect a line of unit length perpendicular to the x- and y-directions are $\rho f(\theta)\cos\theta$ and $\rho f(\theta)\sin\theta$ respectively.

7) The axial force in a fibre can be divided into components along the coordinate axes, $F_{fx} = F_f \cos\theta$, $F_{fy} = F_f \sin\theta$. Now σ_x, σ_y and τ_{xy} can be obtained by integrating over θ. t denotes sheet thickness.

$$
\begin{aligned}
\sigma_x &= \int_0^\pi \frac{\rho}{t} f(\theta) F_f \cos^2\theta d\theta = \\
&= \frac{\rho A_f E_f}{\pi t} \int_0^\pi (\epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta)\cos^2\theta d\theta \\
\sigma_y &= \int_0^\pi \frac{\rho}{t} f(\theta) F_f \sin^2\theta d\theta = \\
&= \frac{\rho A_f E_f}{\pi t} \int_0^\pi (\epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta)\sin^2\theta d\theta \\
\tau_{xy} &= \int_0^\pi \frac{\rho}{t} f(\theta) F_f \sin\theta\cos\theta d\theta = \\
&= \frac{\rho A_f E_f}{\pi t} \int_0^\pi (\epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta)\sin\theta\cos\theta d\theta
\end{aligned}
\tag{3}
$$

By evaluating the following integrals

$$
\begin{aligned}
&\int_0^\pi \cos^4\theta d\theta = \int_0^\pi \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta) d\theta = \frac{1}{8}[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta]_0^\pi = \\
&\qquad = \frac{3\pi}{8} \\
&\int_0^\pi \sin^4\theta d\theta = \int_0^\pi \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta) d\theta = \frac{1}{8}[3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta]_0^\pi = \\
&\qquad = \frac{3\pi}{8} \\
&\int_0^\pi \sin^2\theta\cos^2\theta d\theta = \int_0^\pi \frac{1}{8}(1 - \cos 4\theta) d\theta = \frac{1}{8}[\theta - \frac{1}{4}\sin 4\theta]_0^\pi = \frac{\pi}{8} \\
&\int_0^\pi \sin^3\theta\cos\theta d\theta = \int_0^\pi \frac{1}{8}(2\sin 2\theta - \sin 4\theta) d\theta = \frac{1}{8}[-\cos 2\theta + \frac{1}{4}\cos 4\theta]_0^\pi = 0 \\
&\int_0^\pi \sin\theta\cos^3\theta d\theta = \int_0^\pi \frac{1}{8}(2\sin 2\theta + \sin 4\theta d\theta = \frac{1}{8}[-\cos 2\theta - \frac{1}{4}\cos 4\theta]_0^\pi = 0
\end{aligned}
\tag{4}
$$

we have, on matrix form,

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{\rho A_f E_f}{t} \begin{bmatrix} 3/8 & 1/8 & 0 \\ 1/8 & 3/8 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{5}
$$

The same relation can be obtained by an energy method. The total elastic energy of the network is calculated. The elastic energy stored in a bar of unit length is

$$W_{bar} = \frac{F^2}{2AE} \tag{6}$$

The total elastic energy stored in a unit volume of the network is

$$W = \int_0^{\pi} \frac{\rho f(\theta) F_f^2(\theta)}{2A_f E_f t} d\theta = \frac{\rho A_f E_f}{2\pi t} \int_0^{\pi} (\epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta)^2 d\theta \tag{7}$$

which after integration yields

$$W = \frac{\rho A_f E_f}{t} (\frac{3}{16}\epsilon_x^2 + \frac{2}{16}\epsilon_x \epsilon_y + \frac{3}{16}\epsilon_y^2 + \frac{1}{16}\gamma_{xy}^2) \tag{8}$$

Differentiation with respect to the strain components gives

$$\begin{aligned}
\sigma_x &= \frac{\partial W}{\partial \epsilon_x} = \frac{\rho A_f E_f}{t} (\frac{3}{8}\epsilon_x + \frac{1}{8}\epsilon_y) \\
\sigma_y &= \frac{\partial W}{\partial \epsilon_y} = \frac{\rho A_f E_f}{t} (\frac{1}{8}\epsilon_x + \frac{3}{8}\epsilon_y) \\
\tau_{xy} &= \frac{\partial W}{\partial \gamma_{xy}} = \frac{\rho A_f E_f}{t} (\frac{1}{8}\gamma_{xy}),
\end{aligned} \tag{9}$$

that is, the same as eq. (5).

8) From [1] we have the stress-strain relationship for a homogeneous, linearly elastic, isotropic, continuous material in plane stress.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma \end{bmatrix} \tag{10}$$

Here E denotes Young's modulus and ν denotes Poisson's ratio. Identification of the first two coefficients in the matrices gives

$$\begin{aligned}
\frac{E}{1-\nu^2} &= \frac{\rho A_f E_f}{t} \frac{3}{8} \\
\frac{\nu E}{1-\nu^2} &= \frac{\rho A_f E_f}{t} \frac{1}{8}
\end{aligned} \tag{11}$$

which leads to

$$
\begin{aligned}
E &= \frac{\rho A_f E_f}{3t} \\
\nu &= \frac{1}{3}
\end{aligned} \tag{12}
$$

Since $\dfrac{\rho A_f}{t} = D/d$, D denoting density of sheet [kg/m^3] and d denoting density of cellulose fibre [kg/m^3], an equivalent expression for E in eq. (12) is

$$E = \frac{D}{d}\frac{E_f}{3}, \tag{13}$$

4 Concluding remark

The simple model formulated in the previous section is in principle identical to those of Cox [2] and Campbell [3]. All the references [2]-[8] present different network models that in principle follows the outline in this paper.

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