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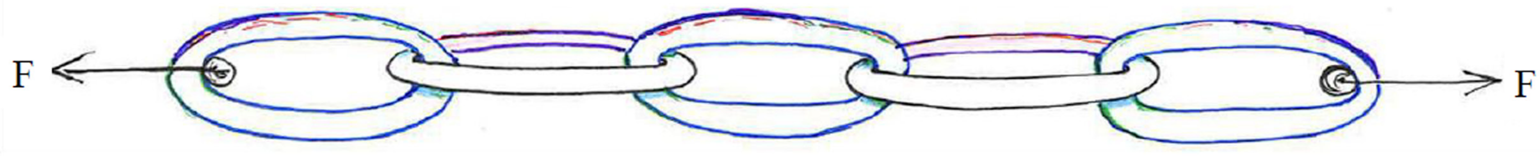

LECTURE NOTES ON SOME PROBABILISTIC STRENGTH CALCULATION MODELS

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DIVISION OF STRUCTURAL MECHANICS

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Lecture notes on some probabilistic strength calculation models

Per Johan Gustafsson, Div. of Structural Mechanics, LU, 2014

- Weibulls distribution function
- Weibulls weakest link model
- Size effects in strength
- Strongest link model

Report TVSM-7161, Division of Structural Mechanics, Lund University, 2014

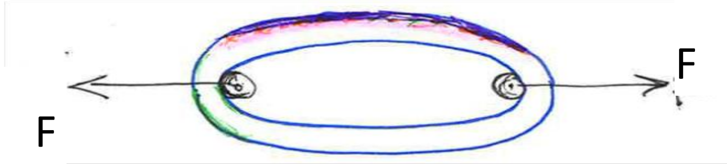
Weibull's distribution and weakest link theory

Table 2.4: A categorization of rational strength analysis approaches. (From (Danielsson, 2013))

		Material strength			
		$f_t = 0$	$f_t =$ finite *deterministic*	$f_t =$ finite *stochastic*	$f_t = \infty$
Fracture energy	$G_f = 0$	-	Conventional Stress Analysis	Weibull Weakest Link Theory	-
	$G_f =$ finite *deterministic*	-	Generalized Linear Elastic and Nonlinear Fracture Mechanics	Probabilistic Gen LE and Nonlinear Fracture Mechanics	Linear Elastic Fracture Mechanics
	$G_f =$ finite *stochastic*	-	Probabilistic Gen LE and Nonlinear Fracture Mechanics	Probabilistic Gen LE and Nonlinear Fracture Mechanics	Probabilistic Linear Elastic Fracture Mechanics
	$G_f = \infty$	-	Ideally Plastic Analysis	Probabilistic Ideally Plastic Analysis	-

The name Weibull*) is internationally associated with:

1) **The Weibull probability distribution function**, is much used in several areas of science and technology for definition of stochastic properties. For instance the stochastic strength properties of a link:

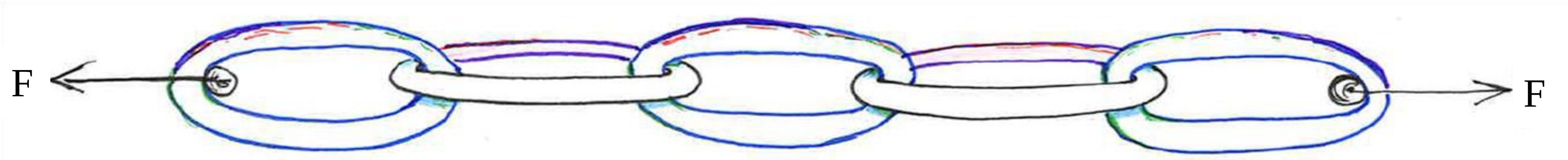


$$S(F) = \begin{cases} 1 - e^{-(F/F_o)^m} & \text{for } F \geq 0 \\ 0 & \text{for } F < 0 \end{cases}$$

- S(F) is the cumulative probability distribution function for the strength of the link. This means that S(F) is the probability that the link will fail if loaded from zero load up to the load F.
- F_o and m are the two parameters that define the properties of the link ("material paramet.").

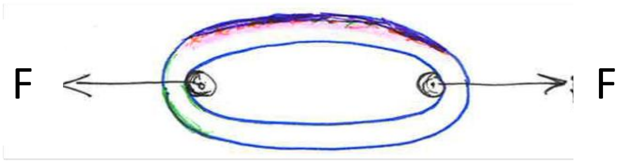
(There is also a three parameter Weibull distribution, not discussed here.)

2) **The Weibull weakest link model,** is used for analysis of the strength of structural elements, e.g. the strength of a chain made up of several links:

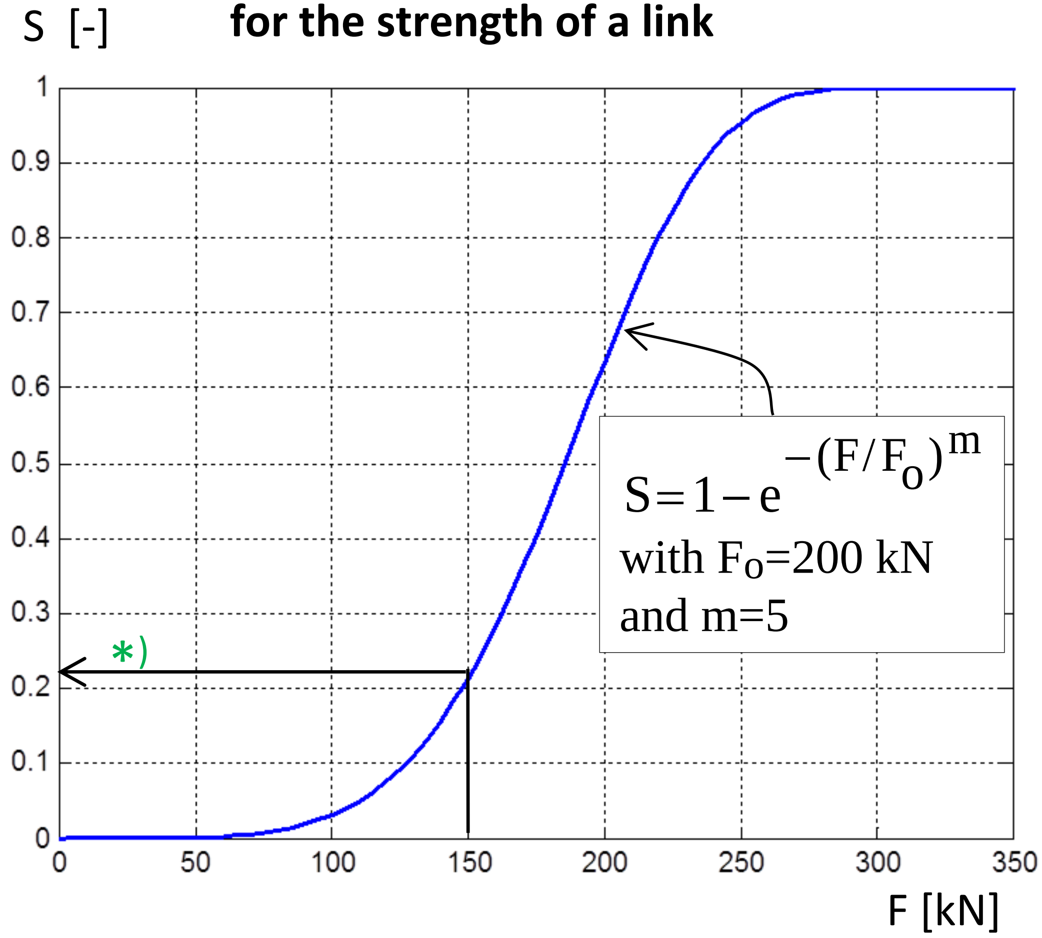


*) Waloddi Weibull, professor at KTH, was born in Kristianstad 1887 and died in France 1979.

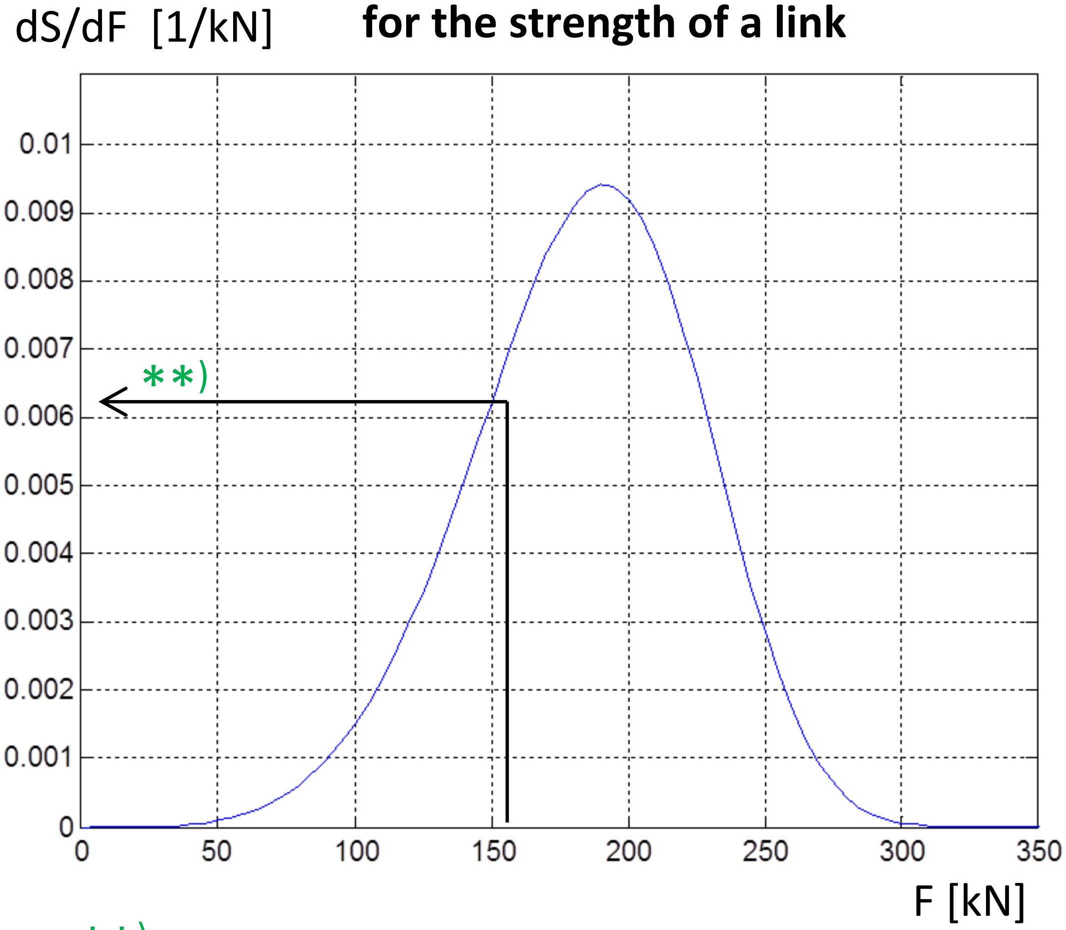
1) The Weibull distribution function *EXAMPLE*



Distribution function
for the strength of a link

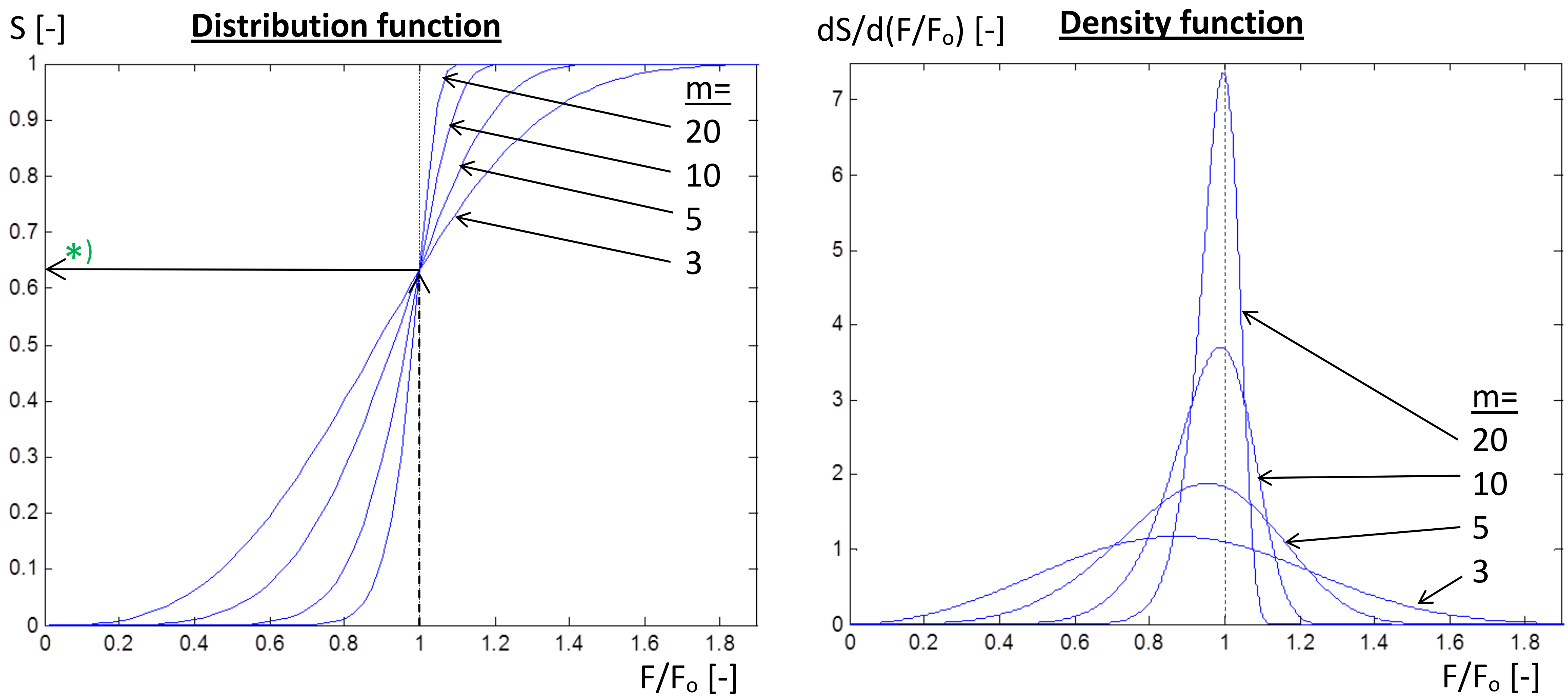


Probabilty density function
for the strength of a link



*)
The reading that the failure probality S=0.21 for F=150 kN means that 21 % of the links have a strength less than or equal to 150 kN.

**)
The reading that the failure probability density dS/dF =0.0061 kN^{-1} for F=150 kN means that 0.61 % of the links fail for each 1 kN increase of the load, when the load is close to 150 kN.



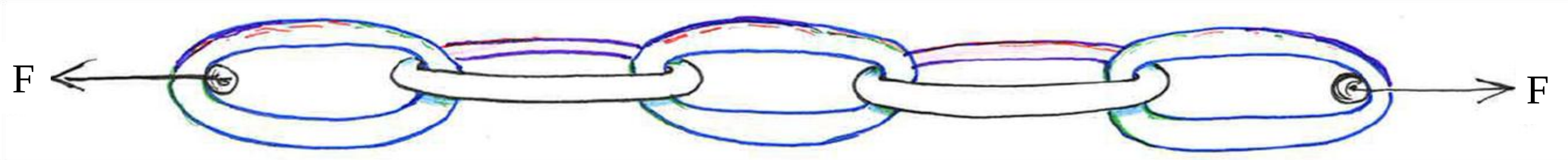
*) The strength scaling parameter F_o is the 63.2 % fractile value.

If F_{mean} and COV are known for some set of test data, then parameters m and F_o of the Weibull distribution can be determined, e.g. from a table or by a Matlab command.

The curve shape parameter m has a one-to-one relation to COV and to normalized fractile and mean values:

m	COV	$F_{5\%}/F_o$	$F_{50\%}/F_o$	F_{mean}/F_o
3	36.4 %	.372	.885	.893
5	22.9 %	.552	.929	.918
10	12.1 %	.743	.964	.951
20	6.3 %	.862	.982	.974

2) The Weibull weakest link model (using Weibulls probality distribution function)



The probability that one specific link fails before the load F is reached: (1)

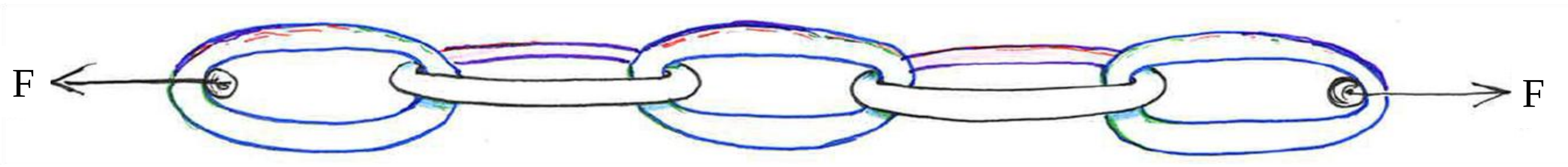
The probability that the link can carry F: $e^{-(F/F_o)^m}$ (2)

The prob. that all n links can carry its load F_i *: $e^{-(F_1/F_o)^m} e^{-(F_2/F_o)^m} \dots \dots e^{-(F_n/F_o)^m} =$

$$= e^{-\sum_{i=1}^{n}(F_i/F_o)^m} \quad (3)$$

The probability that the chain fails before load is F: $1-e^{-\sum_{i=1}^{n}(F_i/F_o)^m}$ (4)

* The load F is allowed to be different for different links.



For the case of equal loading of all n links, F_i=F, is the failure probability:

$$1-e^{-\sum_{i=1}^{n}(F_i/F_o)^m} = 1-e^{-n(F/F_o)^m} = 1-e^{-(F/(F_o/n^{1/m}))^m} \qquad (5)$$

Next the case of different but proportional loading of the n links, $F_i=F_{max}\lambda_i$, where F_{max} is the loading of most loaded link and λ_i defines the load distribution. For this general case is the failure probability:

$$1-e^{-\sum_{i=1}^{n}(F_i/F_o)^m} = 1-e^{-(F_{max}/F_o)^m \sum_{i=1}^{n}\lambda_i^{\,m}} = 1-e^{-(F_{max}/(F_o/(\sum_{i=1}^{n}\lambda_i^{\,m})^{1/m})^m} \qquad (6)$$

Note the similarity between these extreme value distributions and the strength distribution for a single link: $S=1-e^{-(F/F_o)^m}$ … only different scaling of the load! This is a special and convenient feature of Weibull analysis, giving, i.a., equal COV for a link and a chain.

For one link:

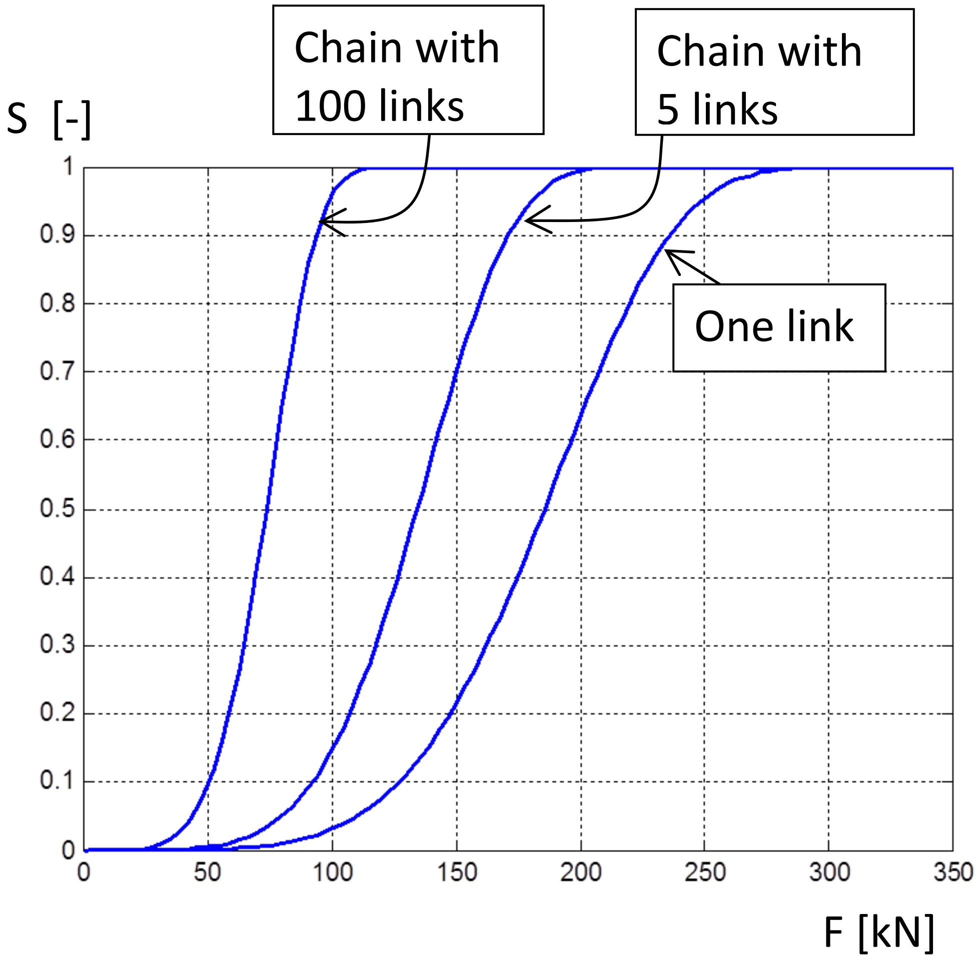
$$S(F) = 1 - e^{-(F/F_o)^m}$$

For a chain with n links:

$$S(F) = 1 - e^{-(F/(F_o/n^{1/m}))^m}$$

Example:

F_o=200 kN
m=5 (COV =21 %)



SUMMARY, for a LINK and a CHAIN

The weibull strength distribution for a single link:

$$S(F) = 1 - e^{-(F/F_o)^m} \qquad (7)$$

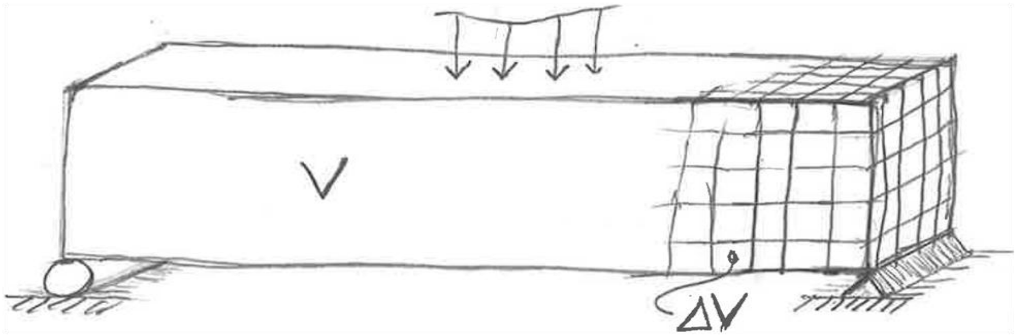
The corresponding extreme value distribution for the strength for a chain made up of n links, where the links can be differently but proportionally loaded according to $F_i = F_{max}\lambda_i$, as quantified by the magnitude of F_{max} then the chain fails,:

$$S(F_{max}) = 1 - e^{-(F_{max}/(F_o/(\sum_{i=1}^{n} \lambda_i{}^m)^{1/m})^m} \qquad (8)$$

Note: the only difference between the two distributions is the load scaling factor:

$$F_o \quad \text{and} \quad F_o/(\sum_{i=1}^{n} \lambda_i{}^m)^{1/m} \quad \text{respectively} \qquad (9)$$

Next the strength of a material volume V made up of small volumes ΔV or dV is analyzed as a chain made up of links.



The volume ΔV_i is loaded by stress σ_i.

The strength properties of the volumes ΔV are defined by parameters σ_o and m.

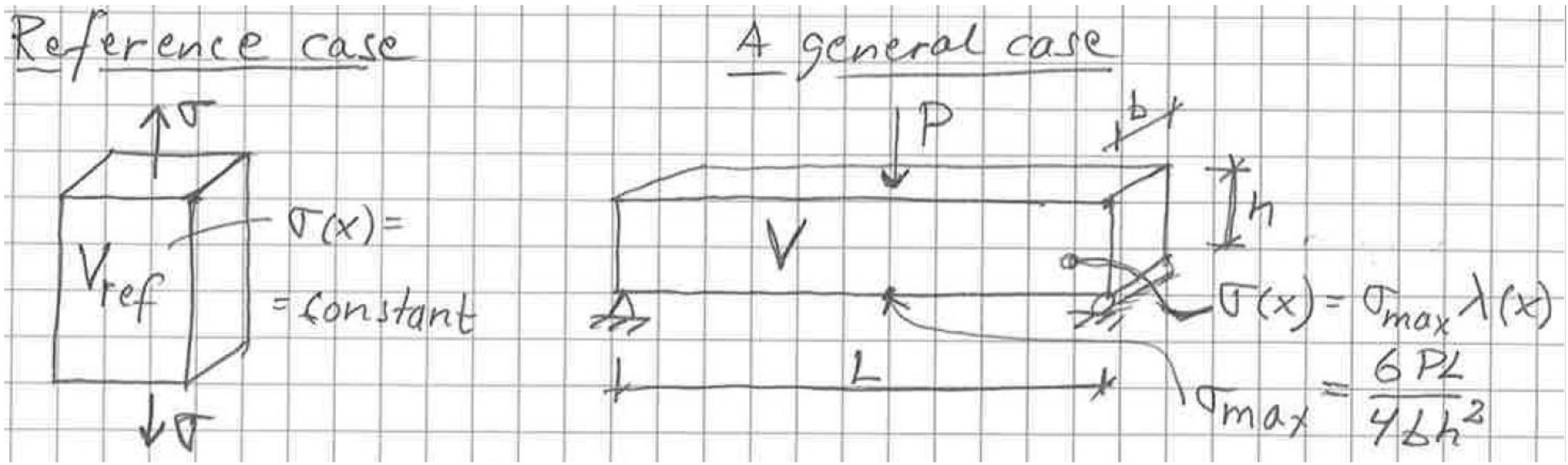
<u>Assumption:</u> *The volume V is assumed to fail as soon as any of volumes ΔV fails. Brittle failure!*

Prob. that V fails: $1-e^{-\sum_{i=1}^{n}(\sigma_i/\sigma_o)^m}$ where n=V/ΔV. For ΔV→dV the sum is:

$$\sum_{i=1}^{V/dV}(\sigma_i/\sigma_o)^m = ((\sigma_i/\sigma_o)^m)_{mean}\frac{V}{dV} = \frac{\frac{\int_V(\sigma(x)/\sigma_o)^m dV}{V}}{V}\frac{V}{dV} = \frac{1}{dV}\int_V(\sigma(x)/\sigma_o)^m dV \quad \ldots(10)$$

Thus the probability of failure of V before the external loading of V has reached a magnitude corresponding to stresses σ(x) in V is

$$S = 1-e^{\frac{-1}{dV}\int_V(\sigma(x)/\sigma_o)^m dV}$$ where σ_o is the 63.2 % fractile strength of the volume dV (11)



The strength distribution for the reference loading case:

$$S = 1 - e^{\frac{-1}{dV}\int_V (\sigma(x)/\sigma_o)^m dV} = 1 - e^{-(\sigma/\sigma_o)^m (V_{ref}/dV)} = 1 - e^{-(\sigma/\sigma_{ref})^m} \quad (12)$$

where σ_{ref} is short for $\sigma_o(dV/V_{ref})^{1/m}$ and is the 63.2 % fractile strength for reference case.

The strength distribution for the general case, with $\sigma_o(dV/V_{ref})^{1/m}=\sigma_{ref}$ and $\sigma(x)=\sigma_{max}\lambda(x)$ is:

$$S = 1 - e^{\frac{-1}{dV}\int_V (\sigma(x)/\sigma_o)^m dV} = 1 - e^{-(\sigma_{max}/\sigma_{ref})^m (1/V_{ref}) \int_V \lambda(x)^m dV} \quad (13)$$

The only difference between the reference loading case strength distribution for σ, (12), and the general loading case strength distribution for σ_{max}, (13), is the load scaling. The ratio between the two loading scaling factors is obtained from eq (12) and (13):

$$\left(\frac{V}{V_{ref}}\right)^{-1/m}\left(\frac{1}{V}\int_V \lambda(x)^m \, dV\right)^{-1/m} \qquad (13)$$

This ratio is valid for any fractile strength value including the median value and also mean value since the distributions are equal when normalized with respect the loading scale factor. For instance is the mean strength for general loading case

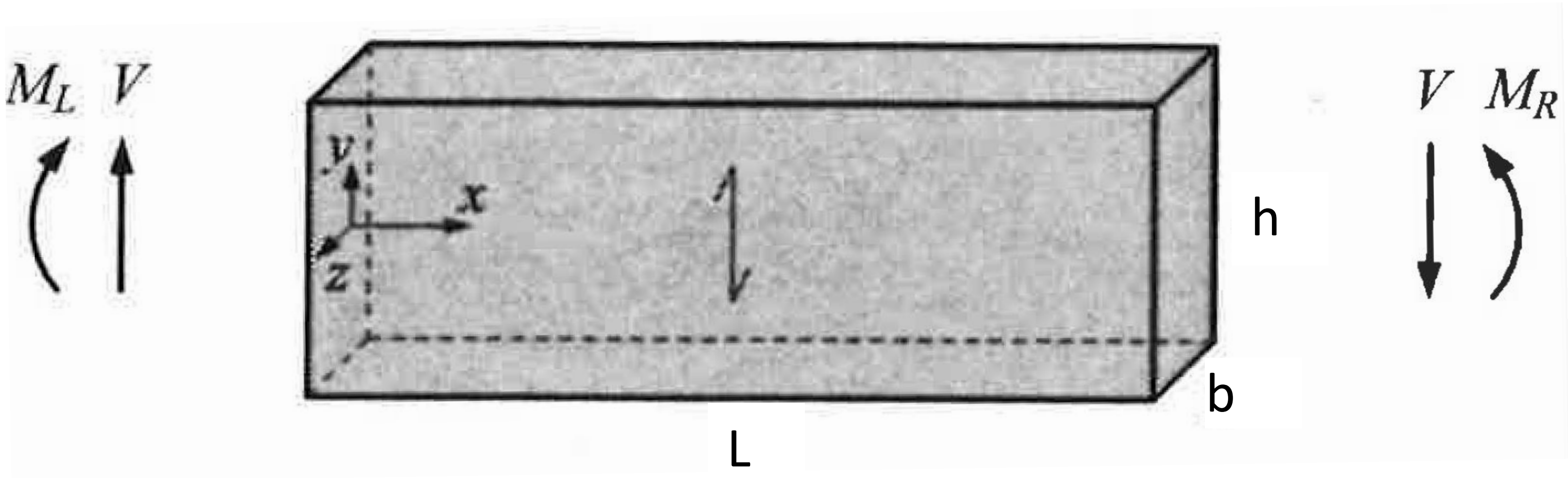
$$\sigma_{max,mean\ failure} = f_t \left(\frac{V}{V_{ref}}\right)^{-1/m}\left(\frac{1}{V}\int_V \lambda(x)^m \, dV\right)^{-1/m} \qquad (14)$$

where f_t is the mean strength of the reference specimen.

- The volume ratio indicates a size effect.
- The λ-integral indicates a stress distribution effect.

The integration can for simple stress distributions be carried out analytically, for other cases numerical integration can be needed.

An example (from Danielsson, 2009)



For constant bending moment M (i.e for M_L=M_R=M and V=0) is the bending strength f_f *) by (14) found to be:

$$f_f = f_t \left(\frac{Lbh}{V_{ref}}\right)^{-1/m} \left(\frac{1}{2(m+1)}\right)^{-1/m}$$

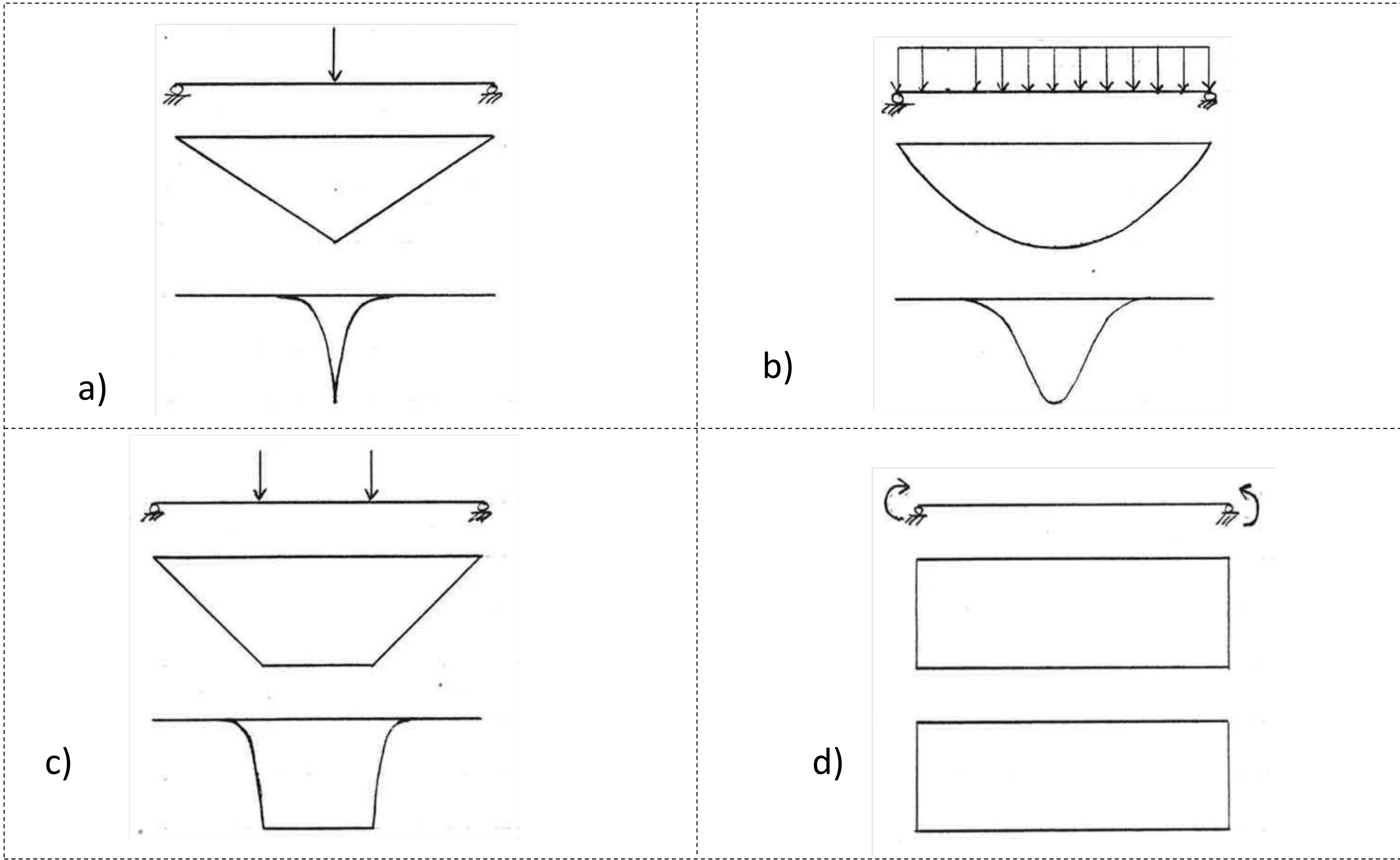
Lbh= V_{ref} and m=10 (corresponding to COV=11.5 %) gives $f_f = 1.35 f_t$

Lbh= $2V_{ref}$ and m=10 (corresponding to COV=11.5 %) gives $f_f = 1.26 f_t$

All (Lbh/V_{ref}) and m→∞ (corresponding to COV=0) gives $f_f = 1.00 f_t$

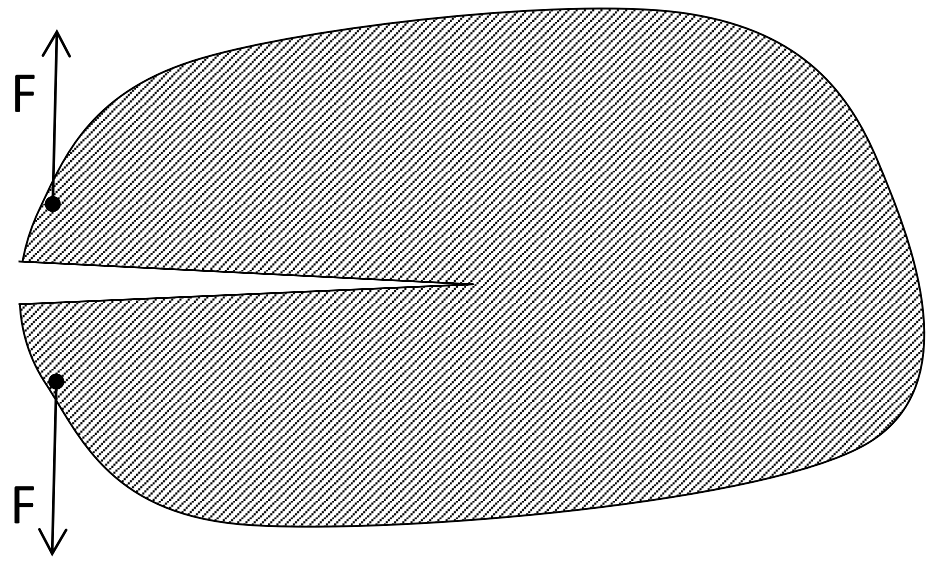
*) the bending strength f_f is by definition $M_{failure}/(bh^2/6)$

By Weibull theory also the failure location probability can be calculated.
Below are for 4 types of loading of a simply supported beam shown the shape of the bending moment diagrams and corresponding failure probability diagrams, valid for m=14 (COV=9.5 %).



The cases a)-d) are ordered according to failure probability at equal max bending moment.
(From Gustafsson, 1983)

Strength at a stress singularity as predicted by Weibull theory



For a sharp crack with a square root stress singularity Weibull theory predicts:

- Zero load carrying capacity if $m>4$ (COV<28.5%)
- Non-zero finite load capacity of the crack tip only for $m=4$ (COV=28.5%)
- Not even for infinitely large load failure at the tip of the crack if $m<4$ (COV>28.5%)

Thus, Weibull theory does not give any useful results when applied to the analysis of cracks and notches with singular stress.

(Details in Gustafsson 1983)

Some concluding remarks relating to the weakest link model

Important assumptions

- Global failure as soon as the failure criterion is fulfilled in one point (brittle!)
- Not applicable to stress singularities (stressed based criterion)
- No strength correlation in between neighboring points of material

Alternatives and extensions

- Integration of the failure probability over volume V replace by surface area A integration
- The 2-parameter strength distribution model replaced by the 3-parameter model (exponent $(\sigma/\sigma_o)^m$ replaced by $((\sigma-\sigma_u)/\sigma_o)^m$)
- The linear elastic model stress-strain model replaced by non-linear model
- Also a “strongest link model” can be proposed and might be useful in some cases.

Applications

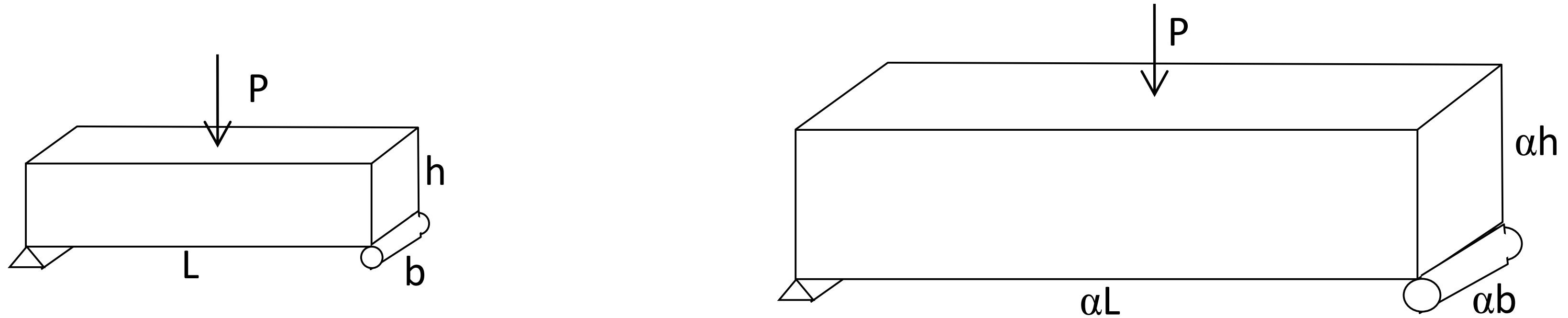
- Glas (surface integration)
- Timber (size effect, effect of bending moment distribution)
- Unreinforced concrete (not successful?)
- Cast iron (?)
- Fatigue of steel structures (?, surface integration)
- Chains ! (?, summation for a finite number possible failure events)
- Structures with several structural elements (?, summation for a finite number of possible failure events)

A load carrying structure with several elements that might fail.

Size effects in strength

Probabalistic strength theories and also fracture mechanics predicts a so-called “size effect” in strength.

What is meant by
“size effect”
in relation to the strength of structural elements?



The influence of structural size on load carrying capacity as predicted by all models based on ”elementary” continuum mechanics *):

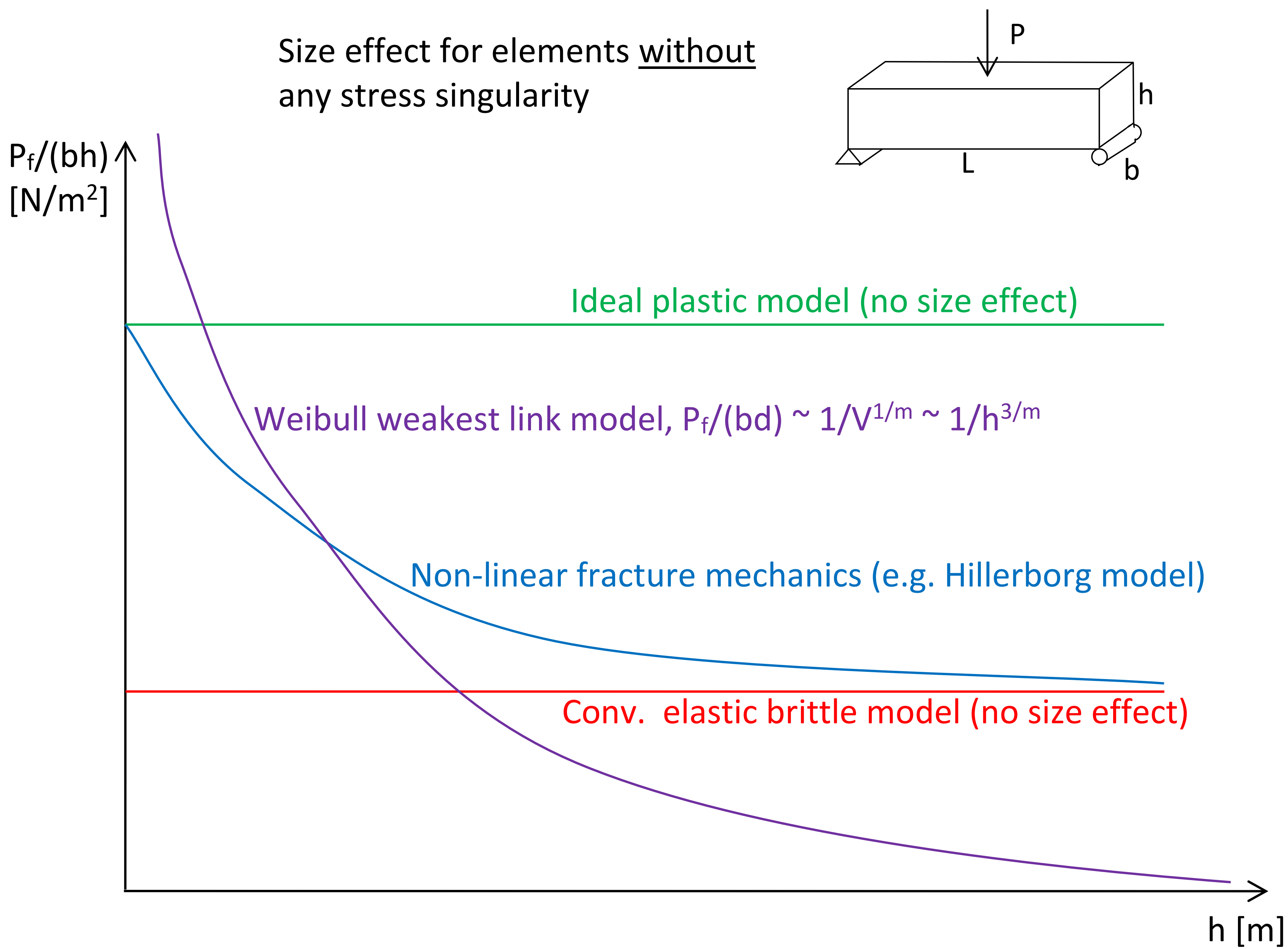
- $P_{\mathbf{failure}} \sim \alpha^2$ (where P is force, e.g. in N)
- $q_{L,\mathbf{failure}} \sim \alpha$ (where q_L is force/length, e.g. in N/m)
- $q_{A,\mathbf{failure}} \sim$ constant (where q_A is force/area, e.g. in N/m^2)
- $q_{V,\mathbf{failure}} \sim 1/\alpha$ (where q_V is force/volume, e.g. in N/m^3)

This is found for all linear and non-linear material and geometrical performance, including stability analysis. The magnitude of displacement at failure is ~ α.

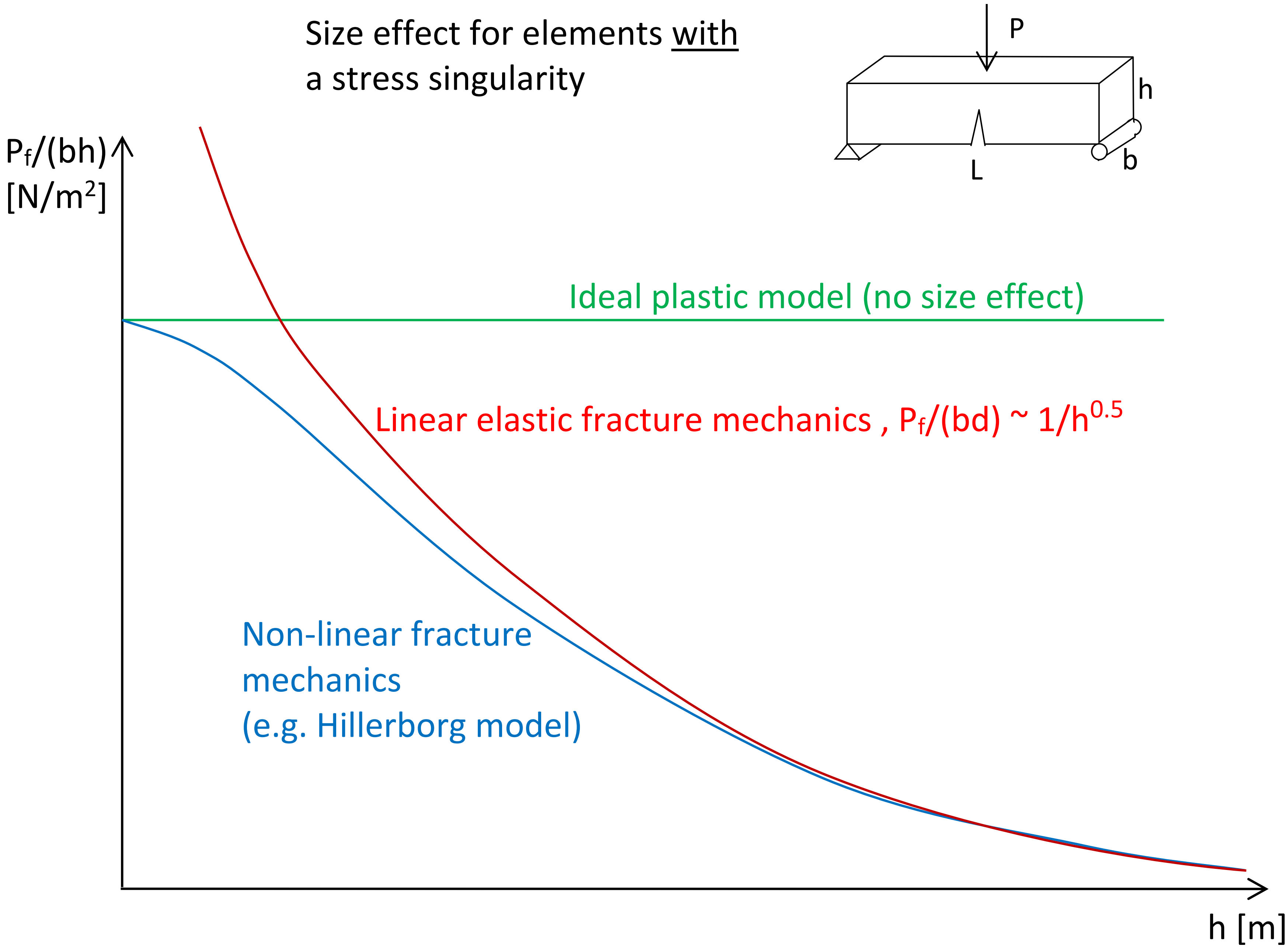
Deviations from the above are referred to as “size effects”.

Size effect are found by: Tests, Fracture mechanics analysis, Weakest link analysis, etc

*) The performance and strength of a homogeneous material is defined in terms of stress and strain.



Size effect for elements without any stress singularity
P
h
L
b
Pf/(bh) [N/m2]
Ideal plastic model (no size effect)
Weibull weakest link model, Pf/(bd) ~ 1/V1/m ~ 1/h3/m
Non-linear fracture mechanics (e.g. Hillerborg model)
Conv. elastic brittle model (no size effect)
h [m]


Size effect for elements with
a stress singularity
P
h
L
b
Pf/(bh)
[N/m2]
Ideal plastic model (no size effect)
Linear elastic fracture mechanics , Pf/(bd) ~ 1/h0.5
Non-linear fracture
mechanics
(e.g. Hillerborg model)
h [m]

Some possible reasons for size effects

- A large volume means greater probability for a weak point (Weibull)
- The size of a FPR is essentially determined by the properties of the material, the size of the FPR is thus not proportional to the global size of the structure (fracture mechanics)
- After the development of strain instability and strain localization is stress governed by absolute material deformation, not by strain (fracture mechanics, Hillerborg model)
- For virtually all materials are the properties close to the surface of a volume more or less different from those within the volume
- etc

Strongest link model

Weibulls weakest link model assumes global failure as soon as some stress based failure criterion is fulfilled for one structural element or in one point. The assumption is clearly valid for the strength of a conventional chain made up of links.

An opposite model and assumption, that of global failure being governed by the strongest link, can be illustrated by the resistance that a zipper gives towards being opened. In this case the zipper link that give the hardest resistance is decisive. Opening of a zipper is analogous to crack propagation. This means that a strongest link model can be relevant in cases where global failure is governed by crack propagation along a crack path of some length, and more generally where failure of two or more structural elements, or points, must occur before global structural failure.

To illustrate a strongest link model the strength of zipper links can for example be assumed according to the Weibull 2-parameter distribution function:

$$S(F) = \begin{cases} 1 - e^{-(F/F_o)^m} & \text{for } F \geq 0 \\ 0 & \text{for } F < 0 \end{cases} \tag{15}$$

where F_o and m are material parameters (link parameters), and where S is the probability that the link fails before the load is F.

Now a zipper with n links is considered. The probability that all links has a strength less than F is S^n. Thus the distribution function for the strength of a zipper with n links is:

$$S(F) = \left(1 - e^{-(F/F_o)^m}\right)^n \tag{16}$$

For comparison, the distribution function for the strength of a conventional chain (as determined by the weakest link model) is:

$$S(F) = 1 - \left(e^{-(F/F_o)^m}\right)^n = 1 - e^{-n(F/F_o)^m} = 1 - e^{-(F/(F_o/n^{1/m}))^m} \tag{17}$$

Comparison between (15) and (17) shows that the weakest link model gives the same shape of the link and the chain distribution functions, only the strength reference value, F_o and $F_o/n^{1/m}$, respectively, is different. This is a special feature of the 2-parameter Weibull distribution function. The strongest link model doesn't show the same convenient feature.

In application of the strongest link model to crack propagation, F and F_o can represent stress intensity K ($=(EG)^{0.5}$) and fracture toughness K_c, respectively, or crack driving force G and critical energy release rate G_c, respectively, or $G^{0.5}$ and $G_c^{0.5}$, respectively, the later alternative giving proportionality between the parameter $G^{0.5}$ and magnitude of external load.

For given load P acting on the structure to be analyzed, K is commonly proportional to P and varies as the crack propagates: $K=PK_1(x)$ or $K_i=PK_1(x_i)$. For a sufficiently short interval $\Delta x=x_{i+1}-x_i$ K can however be regarded as approximately constant, K_i. Crack propagation a distance $n\Delta x$ then gives the strength distribution function

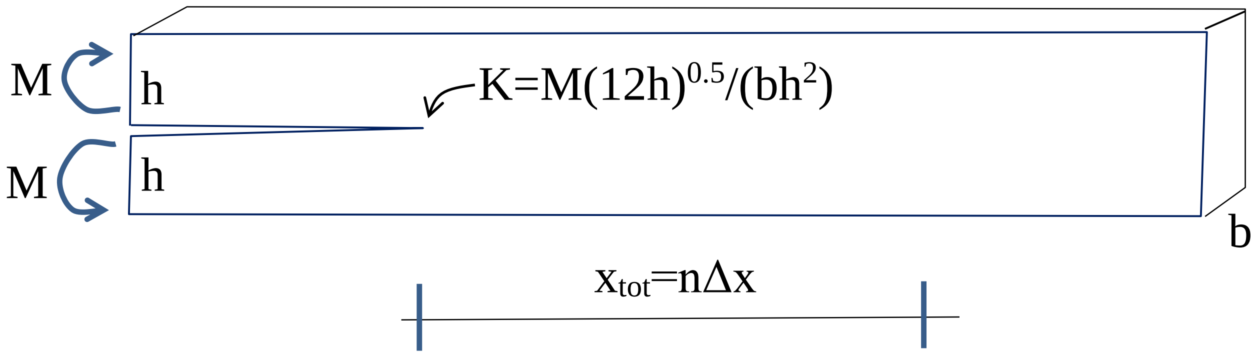
$$S(P)=\prod_{i=1}^{n}\left(1-e^{-(PK_1(x_i)/K_c)^m}\right) \qquad (18)$$

K_c and m are material parameters with length Δx as reference: K_c is the fracture toughness that the material expose during crack propagation the (short) length Δx and m is a measure of the scatter in K_c.

If $K=PK_1(x)$ is constant along the crack propagation path, (18) simplifies to:

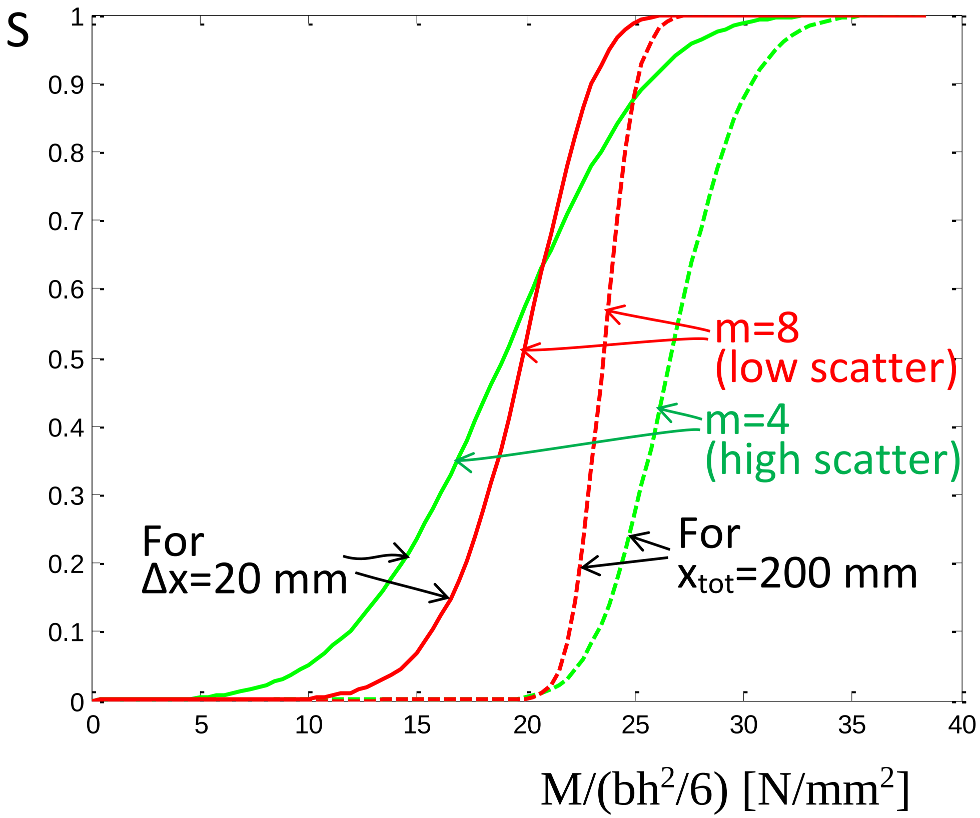
$$S(P) = \left(1 - e^{-(PK_1/K_c)^m}\right)^n \qquad (19)$$

Example 1 (K constant):



$$S(M) = \left(1 - e^{-(K/K_c)^m}\right)^n \qquad (20) \qquad \text{where} \quad K=M(12h)^{0.5}/(bh^2)$$

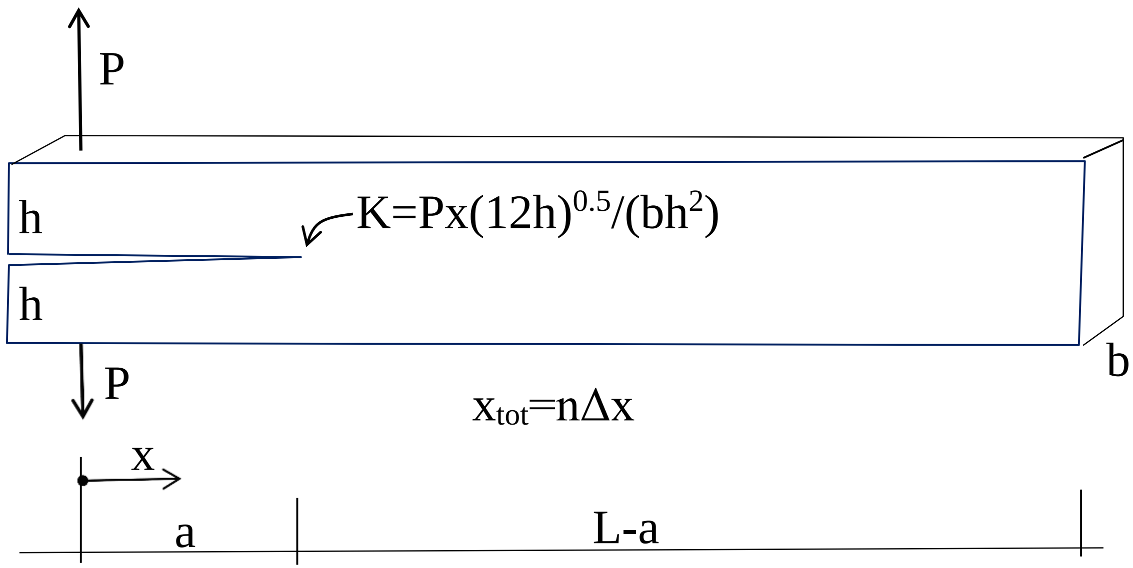
Example 1, cont. For b=h=25 mm and with K_c=70 N/mm$^{3/2}$ for Δx=20 mm (wood) is found:



n	m	5% fractile strength	COV
1	4	9.9	28 %
1	8	14.3	15 %
10	4	22.4	10 %
10	8	21.6	5 %

- Long propagation length before global collaps gives greater strength (reversed size effect).
- Long propagation length can give less scatter in global strength.
- Material strength scatter can be positive in case of crack propagation failure, both for mean strength and 5 % fractile strength.

Example 2 (K varying): Determination of design equation from test results



$$S(P)=\prod_{i=1}^{n}\left(1-e^{-(PK_1(x_i)/K_c)^m}\right) \qquad (21)$$

where $K=PK_1(x_i)=Px_i(12h)^{0.5}/(bh^2)$

Example problem: A large number of tests with b=h=40 mm, a=140 mm and L=300 mm gave for a quality of wood mean strength 1600 N and standard deviation 250 N for P. A 5 % fractile strength design equation for the above type of double cantilever beam is sought.

Calculation: The reference length is more or less arbitrarily taken as Δx=10 mm, giving n=(300-140)/10=16. S(P) and the corresponding mean value and standard deviation of P can then be calculated for x_i=145, 155, … …, 295, for different K_c and m. By simple trial-error calculations K_c = 69.7 N/mm$^{3/2}$ and m= 3.30 is found to correspond to the test recordings for mean and standard deviation.

A 5% fractile strength design equation for double cantilever beams is then

$$0.05 = \prod_{i=1}^{(L\text{-}a)/\Delta x} \left(1 - e^{-(K_i/K_c)^m}\right) \quad (22)$$

from which $P_{0.05}$ can be determined and where Δx=10 mm, K_c = 69.7 N/mm$^{3/2}$, m= 3.30 and $K_i = Px_i(12)^{0.5}/(bh^{1.5})$ with $x_i = a+(i-1/2)\Delta x$.

Eq (22) is simple to solve numerically since the right hand side is 0 for P=0 and then increases continuously with P.

For a=100 mm; b=50 mm and h=75 mm was by (22) $P_{0.05}$ obtained for various L:

L (mm)	110	120	130	140	150	200	400	800
$P_{0.05}$ (N)	2530	3923	4510	4803	4960	5142	5147	5147

It is event that scatter in strength gives a positive length effect. In this example the rate of increase is significant only for small crack propagation lengths (L-a) since K increases with crack length.

In the above examples a simple equation for K has been used, based on beam theory. This simple equation is not very accurate for small (L-a)/h, but makes results regarding influence of the strongest link concept easier to see.

Possible influence and relevant choice of Δx remains to be studied, in particular $\Delta x \rightarrow dx$.