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1 A plasticity model for wood

1.1 Introduction

Although it is possible to construct highly complicated failure criteria, such as the Tsai-Wu criteria, their major drawbacks for practical applications are due to the fact that a large amount of material parameters need to be determined.

In this derivation we restrict the generality such that the following main features of such rather complex criteria are kept:

- orthotropic linear elastic material
- orthotropic plastic behaviour
- different tensile and compressive yield strengths

The aim is to develop a material model for wood that reasonably well describes the behaviour in compression, but remaining simple. The starting point in this derivation is the use of a quadratic failure criterion, following at large the approach in [1].

1.2 Failure criterion

The failure criterion is based on a quadratic expression:

$$\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} + \boldsymbol{\sigma}^T \boldsymbol{q} = 1 \tag{1}$$

where $\boldsymbol{P}$ is a 6×6 matrix containing the strength values, expressed as limit stress in the material directions (diagonal terms) and coupling terms (off-diagonal), and where the coupling terms relate to the possibility of taking into account specific values of bi-axial strength. Different values in tension and compression can be accounted for through the linear term $\boldsymbol{\sigma}^T \boldsymbol{q}$.

For certain values of the components of the off-diagonal terms in $\mathbf{P}$, equation (1) represents a convex, closed surface (ellipsoid) in six-dimensional space. This ellipsoid has its principal axes oriented at different angles to the stress axes. The orientation is determined by the three off-diagonal terms of $\mathbf{P}$. To obtain a convex and closed surface, the following restrictions apply:

$$\begin{aligned} P_{11}P_{22} - P_{12}^2 &\geq 0 \\ P_{11}P_{33} - P_{13}^2 &\geq 0 \\ P_{22}P_{33} - P_{23}^2 &\geq 0 \end{aligned} \tag{2}$$

and, consequently a sufficient criterion would be to choose a diagonal matrix $\boldsymbol{P}$.

For a single closed surface, with a possibility to model the behaviour of an orthotropic material, typically nine independent coefficients of $\boldsymbol{P}$ and the three coefficients of $[q_1\ q_2\ q_3\ 0\ 0\ 0]^T$ are to be defined. This leaves us with twelve material parameters to determine, three tensile strength values, three compressive strength values, three shear strength values and finally, three coupling terms which determine the orientation of the failure surface (its rotation in stress space). The twelve material parameters can be determined by tensile, compressive and shear tests plus an additional three bi-axial tests. If no bi-axial test data is available, the relations in (2) can serve as a help, and thus choosing the terms by using $P_{ij} = f_{ij}\sqrt{P_{ii}P_{jj}}, -1.0 \leq f_{ij} \leq 1.0$ (no sum on i, j).

To model the behaviour of wood in a realistic manner, at least three tensile strength values, three compressive strength values and three shear strength values are needed if perfect plasticity is assumed. In addition, parameters to describe any hardening can of course also be included.

1.3 Equations

1.3.1 Flow rule and plastic multiplier

The current effective yield strength σ_y (which is equal to a constant value of 1.0 for perfect plasticity) is assumed to be a function of an internal variable λ, which is similar (but not identical) to the effective plastic strain. By choosing different functions to describe the dependence of σ_y on λ, different hardening behaviour can be modelled.

We now introduce the yield surface, f, according to:

$$f = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{\sigma}^T\boldsymbol{q} - \sigma_y^2(\lambda) \tag{3}$$

with σ_y being an equivalent yield stress. Differentiating (3) leads to

$$\dot{f} = \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \dot{\boldsymbol{\sigma}} - 2\sigma_y \frac{\partial \sigma_y}{\partial \lambda}\dot{\lambda} \tag{4}$$

Assuming associated plasticity, during plastic loading the consistency relation $\dot{f} = 0$ holds and by use of Drucker's postulate we can write:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{5}$$

with $\dot{\lambda}$ being the time derivative of the plastic multiplier.

1.3.2 Integration of elasto-plastic equations

The total strain are assumed to be decomposed in an elastic part and a plastic part, in incremental notation:

$$\boldsymbol{\Delta\varepsilon} = \boldsymbol{\Delta\varepsilon}^e + \boldsymbol{\Delta\varepsilon}^p \tag{6}$$

Introducing linear elastic behaviour yields

$$\boldsymbol{\Delta\varepsilon}^p = \boldsymbol{\Delta\varepsilon} - \boldsymbol{C}\boldsymbol{\Delta\sigma} \tag{7}$$

with $\boldsymbol{C}$ being the flexibility matrix of the material and $\boldsymbol{\Delta\sigma}$ being the stress increment. The increment in plastic strain is written

$$\boldsymbol{\Delta\varepsilon}^p = \Delta\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{8}$$

The derivative of f with respect to $\boldsymbol{\sigma}$ is

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{q} \tag{9}$$

We now rewrite (8) using (7) and (9) to obtain

$$\begin{aligned} \boldsymbol{C}\boldsymbol{\Delta\sigma} - \boldsymbol{\Delta\varepsilon} + \Delta\lambda \left(\boldsymbol{P}\left(\boldsymbol{\sigma}_0 + \boldsymbol{\Delta\sigma}\right) + \boldsymbol{q}\right) &= \\ = \left(\boldsymbol{C} + \Delta\lambda\boldsymbol{P}\right)\left(\boldsymbol{\sigma}_0 + \boldsymbol{\Delta\sigma}\right) - \left(\boldsymbol{\varepsilon}^e + \boldsymbol{\Delta\varepsilon} - \Delta\lambda\boldsymbol{q}\right) &= 0 \end{aligned} \tag{10}$$

with $\boldsymbol{\sigma}_0$ being the stress at the beginning of the load step. The stress at the end of the load step is given by

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_0 + \boldsymbol{\Delta\sigma} = \left(\boldsymbol{C} + \Delta\lambda\boldsymbol{P}\right)^{-1}\left(\boldsymbol{\varepsilon}^e + \boldsymbol{\Delta\varepsilon} - \Delta\lambda\boldsymbol{q}\right) \tag{11}$$

which can be interpreted as an elastic predictor-plastic corrector algorithm.

Now introducing(11) into the yield function (3) results in the yield condition as a function of the plastic multiplier $\Delta\lambda$. To solve for the unknown plastic multiplier using e.g. a Newton-Raphson iteration scheme, we need the derivative of (3) with respect to $\Delta\lambda$:

$$\frac{\partial f}{\partial \Delta\lambda} = \left(\frac{\partial f}{\partial \boldsymbol{\sigma}_n}\right)^T \frac{\partial \boldsymbol{\sigma}_n}{\partial \Delta\lambda} - 2\sigma_y \frac{\partial \sigma_y}{\partial \Delta\lambda} \tag{12}$$

noting that for a matrix $\boldsymbol{A}^{-1}$ we have $\left(\boldsymbol{A}^{-1}\right)' = \boldsymbol{A}^{-1}\boldsymbol{A}'\boldsymbol{A}^{-1}$ we obtain the derivative of (11) as:

$$\frac{\partial \boldsymbol{\sigma}_n}{\partial \Delta\lambda} = -\left(\boldsymbol{C} + \Delta\lambda 2\boldsymbol{P}\right)^{-1}\left(\boldsymbol{P}\left(\boldsymbol{C} + \Delta\lambda\boldsymbol{P}\right)^{-1}\left(\boldsymbol{\varepsilon}^e + \boldsymbol{\Delta\varepsilon} - \Delta\lambda\boldsymbol{q}\right) - \boldsymbol{q}\right) \tag{13}$$

Using (9) We finally obtain for the derivative of the yield function:

$$\begin{aligned} \frac{\partial f}{\partial \Delta\lambda} &= -\left(\boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{q}\right) \times \left(\boldsymbol{C} + \Delta\lambda\boldsymbol{P}\right)^{-1}\left[\boldsymbol{P}\left(\boldsymbol{C} + \Delta\lambda\boldsymbol{P}\right)^{-1}\left(\boldsymbol{\varepsilon}^e + \boldsymbol{\Delta\varepsilon} - \Delta\lambda\boldsymbol{q}\right) - \boldsymbol{q}\right] \ldots \\ &\ldots - 2\sigma_y \frac{\partial \sigma_y}{\partial \Delta\lambda} \end{aligned} \tag{14}$$

1.3.3 Tangent stiffness operators

The total strain at the end of an iteration, i, starting from the previous converged state with strains $\boldsymbol{\varepsilon}^n$ is given by

$$\boldsymbol{\varepsilon}^i = \boldsymbol{\varepsilon}^n + \boldsymbol{\Delta\varepsilon^{i,e}} + \boldsymbol{\Delta\varepsilon^{i,p}} \tag{15}$$

Introducing the relations for the incremental strain:

$$\boldsymbol{\Delta\varepsilon}^{i,e} = \boldsymbol{C}\left(\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^n\right) \tag{16}$$

and

$$\boldsymbol{\Delta\varepsilon}^{i,p} = \Delta\lambda^i \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{17}$$

we obtain

$$\boldsymbol{\varepsilon}^i = \boldsymbol{\varepsilon}^n + \boldsymbol{C}\left(\boldsymbol{\sigma}^i - \boldsymbol{\sigma}^n\right) + \Delta\lambda^i \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{18}$$

The time derivative of this results in:

$$\dot{\boldsymbol{\varepsilon}}^i = \boldsymbol{C}\dot{\boldsymbol{\sigma}}^i + \Delta\lambda^i \frac{\partial^2 f}{\partial \boldsymbol{\sigma}^2}\dot{\boldsymbol{\sigma}}^i + \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{19}$$

For incremental loading steps the second term in (19) vanishes, but for finite loading steps, it can contribute considerably to the elasto-plastic tangent stiffness. Define the matrix $\boldsymbol{H}$ according to:

$$\boldsymbol{H} = \boldsymbol{C} + \Delta\lambda^i \frac{\partial^2 f}{\partial \boldsymbol{\sigma}^2} \tag{20}$$

so that equation (19) can be written

$$\dot{\boldsymbol{\varepsilon}}^i = \boldsymbol{H}\dot{\boldsymbol{\sigma}}^i + \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{21}$$

premultiplication of (21) with $\boldsymbol{H}^{-1}$ and $\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T$ and rearranging results in

$$\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \dot{\boldsymbol{\sigma}}^i = \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}\dot{\boldsymbol{\varepsilon}}^i - \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}\dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = 2\sigma_y \frac{\partial \sigma_y}{\partial \lambda}\dot{\lambda} \tag{22}$$

where the consistency relation (4) was invoked. Thus the plastic multiplier is expressed using:

$$\dot{\lambda} = \frac{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}\dot{\boldsymbol{\varepsilon}}^i}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}\frac{\partial f}{\partial \boldsymbol{\sigma}} + 2\sigma_y\dot{\sigma}_y} \tag{23}$$

Now making use again of (21) premultiplied by $\boldsymbol{H^{-1}}$ and inserting (23) finally yields:

$$\dot{\boldsymbol{\sigma}}^i = \boldsymbol{H}^{-1}\dot{\boldsymbol{\varepsilon}}^i - \frac{\boldsymbol{H}^{-1}\frac{\partial f}{\partial \boldsymbol{\sigma}}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{H}^{-1}\frac{\partial f}{\partial \boldsymbol{\sigma}} + 2\sigma_y\dot{\sigma}_y}\,\dot{\boldsymbol{\varepsilon}}^i \tag{24}$$

Or with obvious notation for the elasto-plastic stiffness matrix $\boldsymbol{D}^{ep}$:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\varepsilon}} \tag{25}$$

1.4 Choice of specific yield surface

Until now, we left unspecified the explicit construct of matrices $\boldsymbol{P}$ and $\boldsymbol{q}$. In [1] the so-called Hoffman criterion is used and explicit equations for the components of $\boldsymbol{p}$ and $\boldsymbol{q}$ are given. Due to the high degree of anisotropy of wood, it is sometimes not possible to fit a convex, closed yield surface to test data, due to the restrictions given in (2).

A special type of yield surface that does not have such restrictions can instead be constructed from a number of (hyper)ellipsoids, each active i a certain part of the stress space only. In a two-dimensional case, this can be described by letting the yield surface consist of four different ellipses, each with their centre in the origin, and defined such that a smooth and continuous surface is obtained, see Figure 1 (left). In such a case the yield surface can be simplified to:

$$f = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - \sigma_y^2 = 0 \tag{26}$$

with the matrix P being diagonal and containing the yield stress for the different directions in that diagonal. As an example $P_{22} = \frac{2}{f_{22}^2}$ with f_{22} denoting the strength of the material in its material direction 2 (compression or tension). As the yield surface is built up from several separate but centered ellipsoids, continuity and smoothness of the surface is guaranteed. The strength values corresponding to the direct stress components are thus chosen depending on the current state of stress when evaluating the failure criterion, such that *e.g.* the compression strength is used for $\sigma_{22} < 0$.

This approach has one big advantage. To explain the advantage, we must first look at the general case, in two dimensions for simplicity. The normal to the failure surface indicates the direction of the increment in plastic strains at plastic loading for associated plasticity. At the points where the ellipse intersects the axes, we then would need to have the plastic strain increments to be in line with what is physically reasonable. For example, at uniaxial straining in the x-direction, we would expect the strain increments to be positive in the x-direction and negative in the other two directions. This places some restrictions on the ellipse to use for the failure surface.

In Figure 1 (right), an ellipse is drawn. This ellipse fulfils some basic characteristics, such as higher tensile strength than compressive strength in the x_1-direction, but the opposite in the x_2-direction. However, as indicated in the figure, the plastic straining at uniaxial loading in the x_1-direction, would lead to plastic strain increments of non-physical nature. In the case indicated, the plastic strain increment would be positive in both the x_1 and the x_2 directions.

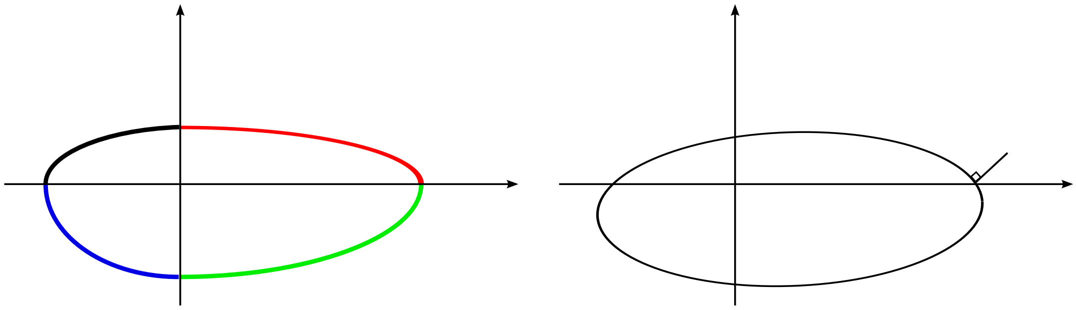

Figure 1: Left: A possible failure surface in two dimensions. Right:A failure surface with non-physical behaviour in plastic straining.

1.5 Choice of hardening function

The currently implemented hardening function was chosen so as to give a (piecewise) linear hardening in uniaxial loading for the specific choice of yield surface in line with the above-mentioned multi-surface approach. Three piecewise linear intervals were implemented, thus making possible to include e.g. the typical densification seen in compression perpendicular to the grain. The chosen hardening functions are:

$$\sigma_y = \begin{cases} e^{(k_1\lambda)} & \lambda < \lambda_1 \\ e^{(k_1\lambda_1)} \cdot e^{(k_2(\lambda-\lambda_1))} & \lambda_1 \leq \lambda < \lambda_2 \\ e^{(k_1\lambda_1)} \cdot e^{(k_2(\lambda_2-\lambda_1))} \cdot e^{(k_3(\lambda-\lambda_2))} & \lambda \geq \lambda_2 \end{cases} \tag{27}$$

To verify that this gives indeed a linear hardening, consider the yield surface (26) and insert the initial hardening function from (27), valid for $\lambda < \lambda_1$

$$f = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - e^{(k_1\lambda)} = 0 \tag{28}$$

which, with the traditional definition of the plastic strain rate, $\boldsymbol{\varepsilon}^p$, would result in

$$\dot{\boldsymbol{\varepsilon}^p} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\boldsymbol{P}\boldsymbol{\sigma} \tag{29}$$

Looking now at uniaxial loading, we have for example:

$$\dot{\varepsilon}^p_{11} = \dot{\lambda} P_{11}\sigma_{11} \text{ with } P_{11} = \frac{2}{f_c^2} \tag{30}$$

With the assumed development of the relative yield stress, see (27), we have $\sigma_{11} = f_c\sigma_y = f_c e^{k_1\lambda}$ and thus:

$$\dot{\varepsilon}^p_{11} = \dot{\lambda}\frac{2e^{k_1\lambda}}{f_c} \tag{31}$$

Integrating (31) leads to

$$\varepsilon^p_{11} = \frac{2}{f_c k_1}e^{k_1\lambda} + C \tag{32}$$

Where the integration constant, C, is chosen to fulfil

$$\varepsilon^p(\lambda = 0) = 0. \tag{33}$$

Which results in

$$C = -\frac{2}{f_c k_1} \tag{34}$$

We then obtain

$$\varepsilon^p_{11} = \frac{2}{f_c k_1}e^{k_1\lambda} - \frac{2}{f_c k_1} \tag{35}$$

which can be expressed in terms of a linear function in ε^p_{11} as:

$$\varepsilon^p_{11}\frac{k_1 f_c^2}{2} + f_c = f_c e^{k_1\lambda}(= f_c\sigma_y = \sigma_{11}) \tag{36}$$

Thus, the chosen exponential function gives a (piecewise) linear hardening in ε^p at uniaxial loading. The corresponding plastic modulus, H, is given by

$$H = \frac{k_1 f_c^2}{2}. \tag{37}$$

which then holds for $(\lambda \leq \lambda_1)$. The limit value of λ_1 corresponds to the plastic strain $\varepsilon_{p,11}$ according to

$$\varepsilon^p_{11,\lambda_1} = \frac{2}{f_c k_1}\left(e^{k_1\lambda_1} - 1\right) \tag{38}$$

For any other interval of the piecewise defined expression for σ_y, the derivation resulting in the above equations is relatively straight forward.

References

[1] Schellekens, J.C.J and de Borst, R. *The use of the Hoffman yield criterion in finite element analysis of anisotropic composites* Computers and Structures 37(6), pp. 1087–1096, 1990.